EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet **Discussion 7A**

The following notes are useful for this discussion: Note 10 and Note 11.

**1. Changing behavior through feedback**

In this question, we discuss how feedback control can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i+1] = 0.9x[i] + u[i] + w[i] \tag{1}$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time $i$.

**Is the system stable? If $|w[i]| \le \epsilon$, what can you say about $|x[i]|$ at all times $i$ if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?**

**Solution:** The system is stable, as $\lambda = 0.9 \to |\lambda| < 1$. We can say that $|x[i]|$ is bounded at all time if the disturbance is bounded. Unrolling the system's recursion and extrapolating the general form,

$$x[0] = 0 \tag{2}$$
$$x[1] = w[0] \tag{3}$$
$$x[2] = 0.9w[0] + w[1] \tag{4}$$
$$x[3] = 0.9^2 w[0] + 0.9w[1] + w[2] \tag{5}$$
$$\vdots \tag{6}$$
$$x[i] = \sum_{k=0}^{i-1} 0.9^{i-k-1} w[k]. \tag{7}$$

We can check that this form works by plugging it into our recursion:

$$x[i+1] = 0.9x[i] + w[i] = 0.9\left(\sum_{k=0}^{i-1} 0.9^{i-k-1} w[k]\right) + w[i] = \sum_{k=0}^{i-1} 0.9^{i-k} w[k] + w[i] = \sum_{k=0}^{i} 0.9^{i-k} w[k] \tag{8}$$

which is exactly what our formula predicts.

Thus

$$|x[i]| = \left|\sum_{k=0}^{i-1} 0.9^{i-k-1} w[k]\right| \le \sum_{k=0}^{i-1} \left|0.9^{i-k-1} w[k]\right| = \sum_{k=0}^{i-1} 0.9^{i-k-1}\epsilon. \tag{9}$$

In the limit as $i \to \infty$, by the geometric series formula,

 

$$|x[i]| \leq \frac{\epsilon}{1-0.9} = 10\epsilon \tag{10}$$

(b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. **For what values of $\lambda$ can you get the system to behave like:**

$$x[i+1] = \lambda x[i] + w[i]? \tag{11}$$

**How would you pick $f$?**

(*Note*: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.)

**Solution:** We can control the system to have any value of $\lambda$, as long as we're not limited on the values of $f$.

$$x[i+1] = 0.9x[i] + fx[i] + w[i] = \lambda x[i] + w[i]. \tag{12}$$

Fitting terms, $f = \lambda - 0.9$. Note we can get a $\lambda > 1$ if we so desire; there is nothing stopping us from putting arbitrarily big/small $\lambda$ by the choice of $f$.

(c) **For the previous part, which $f$ would you choose to minimize how big $|x[i]|$ can get?**

**Solution:** From eq. (11), in order to have the minimum bound on $|x[i]|$, $\lambda = 0$. To get this $\lambda$, $f = -0.9$. In the limit as $i \to \infty$ in this case,

$$|x[i]| \leq \frac{\epsilon}{1-0} = \epsilon \tag{13}$$

The minimum bound on $|x(i)| = \epsilon$ is the same bound as on the disturbance.

(d) **What if instead of a $0.9$, we had a $3$ in the original eq. (1). What, if anything, would change?**

**Solution:** If our system were now,

$$x[i+1] = 3x[i] + u[i] + w[i], \tag{14}$$

the system would no longer be stable. However, we can still choose any eigenvalue $\lambda$ using closed loop feedback. In this case, $f = \lambda - 3$.

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{15}$$

where we further assume that $B$ is an invertible square matrix. Futher, suppose we decide to apply linear feedback control using a square matrix $F$ so we choose $\vec{u}[i] = F\vec{x}[i]$.

**For what values of matrix $G$ can you get the system to behave like:**

$$\vec{x}[i+1] = G\vec{x}[i] + \vec{w}[i]? \tag{16}$$

**How would you pick $F$ given knowledge of $A, B$ and the desired goal dynamics $G$?**

**Solution:** Since in this case our input is the same rank as our output, we can arbitrarily choose the matrix $G$. As long as $B$ is invertible (as given), we can define:

$$\begin{aligned}
\vec{x}[i+1] &= A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] && (17)\\
&= A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i] && (18)\\
&= (A+BF)\vec{x}[i] + \vec{w}[i] && (19)\\
&= G\vec{x}[i] + \vec{w}[i] && (20)
\end{aligned}$$

Therefore, matching terms,

$$A + BF = G \implies F = B^{-1}(G - A). \tag{21}$$

  

**2. Controlling states by designing sequences of inputs**

This is something that you saw in 16A in the Segway problem. In that problem, you were given a semi-realistic model for a segway. Here, we are just going to consider the following matrix chosen for ease of understanding what is going on:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]. \tag{23}$$

(a) We are given the initial condition $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $\ell \geq 0$. We don't need to stay there, we just want to be in this state at that time. **What is the smallest $\ell$ such that this is possible? What is our choice of sequence of inputs $u[i]$?**

**Solution:** To ease notation, let

$$\vec{x}[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \\ x_3[i] \\ x_4[i] \end{bmatrix}. \tag{24}$$

Writing out expressions for $x[i]$ we get:

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix}, \tag{25}$$

$$\vec{x}[2] = \begin{bmatrix} x_3[0] \\ x_4[0] \\ u[0] \\ u[1] \end{bmatrix}, \qquad \vec{x}[3] = \begin{bmatrix} x_4[0] \\ u[0] \\ u[1] \\ u[2] \end{bmatrix}, \tag{26}$$

and if $i \geq 4$,

$$\vec{x}[i] = \begin{bmatrix} u[i-4] \\ u[i-3] \\ u[i-2] \\ u[i-1] \end{bmatrix}. \tag{27}$$

  

Hence, the smallest $\ell$ is equal to 4, with $u[0] = [1]$, $u[1] = [2]$, $u[2] = [3]$, $u[3] = [4]$.

(b) **What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest $\ell$ and what is our choice of $u[i]$?**

**Solution:** Looking over our expressions for x[i] from the previous part, we see that the earliest $\ell$ whose expression can be set to the desired state is $\ell = 1$ requiring $u[0] = 4$.

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ u[0] \end{bmatrix}. \tag{28}$$

(c) **If we start from $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, what is smallest $\ell$ such that $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, what is corresponding $u[i]$?**

**Solution:** Looking over our expressions for x[i], we see that the earliest $\ell$ whose expression can be set to the desired state in this case is $\ell = 4$ requiring $u[0] = 1$, $u[1] = 2$, $u[2] = 3$, $u[3] = 4$.

(d) **If you would like to make sure that at time $\ell$ we are at $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ for the state, what controls could you use to get there? How big does $\ell$ have to be for this strategy to work?**

**Solution:** As you might notice, using inputs $u[i] \in \{a, b, c, d\}$ (in that order), we can get to any desired state $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Hence with $\ell = 4$, we can guarantee that the $\vec{x}[\ell]$ is our desired state.

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