EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet **Discussion 8A**

In this discussion we review problems from Spring 2021, Midterm.

1. **Honor Code**

2. **Pre-Examination**

3. **Potpourri!**

(a) **[4 points]** You are given the graph in Figure 1. **Express the coordinates of vectors $\vec{v}$ and $\vec{w}$ in both Cartesian ((x, y)) and Polar $\left(re^{j\theta}\right)$ form**. *You do not need to show your work for this subpart.*
You may use the $\operatorname{atan2}$ or $\tan^{-1}$ function for angle (θ) as necessary.

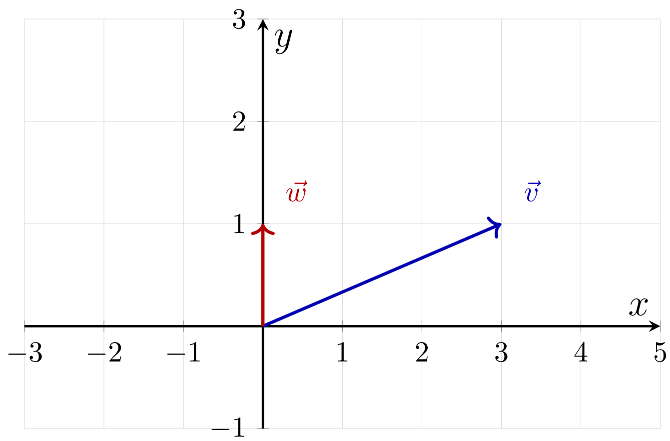


Figure 1: Vectors in the $\mathbf{x} - \mathbf{y}$ plane

i. **Label $\vec{v}$ with its corresponding Cartesian ((x, y)) and Polar $\left(re^{j\theta}\right)$ coordinates, in the given form.**

Solution:

Vector $\vec{v}$ Cartesian = $(3, 1)$

Vector $\vec{v}$ Polar = $\sqrt{10}e^{j\,\operatorname{atan2}(1,3)} \equiv \sqrt{10}e^{j\tan^{-1}\left(\frac{1}{3}\right)}$

ii. **Label $\vec{w}$ with its corresponding Cartesian ((x, y)) and Polar $\left(re^{j\theta}\right)$ coordinates, in the given form.**

Solution:

Vector $\vec{w}$ Cartesian = $(0, 1)$

Vector $\vec{w}$ Polar $=1e^{j\frac{\pi}{2}} \equiv -1e^{-j\frac{\pi}{2}} \equiv 1e^{j\frac{-3\pi}{2}} \equiv -1e^{j\frac{3\pi}{2}}$

(b) **[6 points]** You are given an input voltage signal below:

$$v_{\text{in}}(t) = -2\cos\left(\omega t + \frac{\pi}{3}\right). \tag{1}$$

Convert the signal of eq. (1) to its phasor representation. That is, find $\widetilde{V}_{\text{in}}$. *Justify your answer.*

Solution:

$$\boxed{\widetilde{V}_{\text{in}} = -e^{j\frac{\pi}{3}}}$$

We can use Euler's formulas here, which states that:

$$\cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$$
$$\sin(x) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$$

Applying the first of these formulas and simplifying $v_{\text{in}}(t)$:

$$\begin{aligned}
v_{\text{in}}(t) &= -2\cos\left(\omega t + \frac{\pi}{3}\right) && (2)\\
&= -2 \cdot \frac{1}{2}\left(e^{j\left(\omega t + \frac{\pi}{3}\right)} + e^{-j\left(\omega t + \frac{\pi}{3}\right)}\right) && (3)\\
&= -1\left(e^{j\omega t}e^{j\frac{\pi}{3}} + e^{-j\omega t}e^{-j\frac{\pi}{3}}\right) && (4)\\
&= -e^{j\omega t}e^{j\frac{\pi}{3}} - e^{-j\omega t}e^{-j\frac{\pi}{3}} && (5)\\
&= \left(-e^{j\frac{\pi}{3}}\right)e^{j\omega t} + \left(-e^{-j\frac{\pi}{3}}\right)e^{-j\omega t} && (6)
\end{aligned}$$

When we have a term of the form $u(t) = \widetilde{U}e^{j\omega t} + \overline{\widetilde{U}}e^{-j\omega t}$, we denote $\widetilde{U}$ as the phasor for the time-domain signal. So, we apply this logic here:

$$\widetilde{V}_{\text{in}} = -e^{j\frac{\pi}{3}} \tag{7}$$

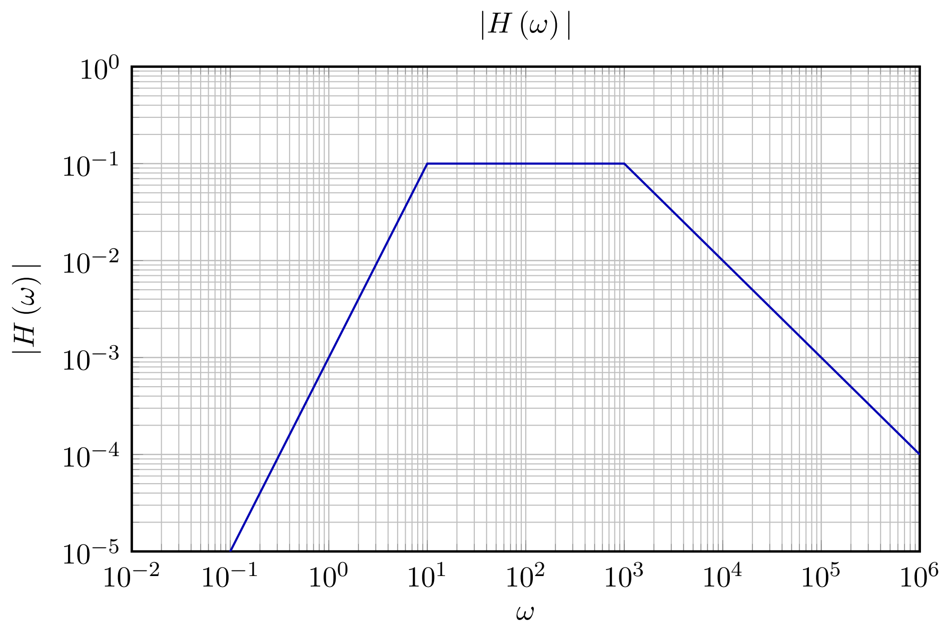
Alternatively,

$$\widetilde{V}_{\text{in}} = -e^{j\frac{\pi}{3}} = e^{\mathrm{j}\pi}e^{j\frac{\pi}{3}} = e^{\mathrm{j}\frac{4\pi}{3}} \tag{8}$$

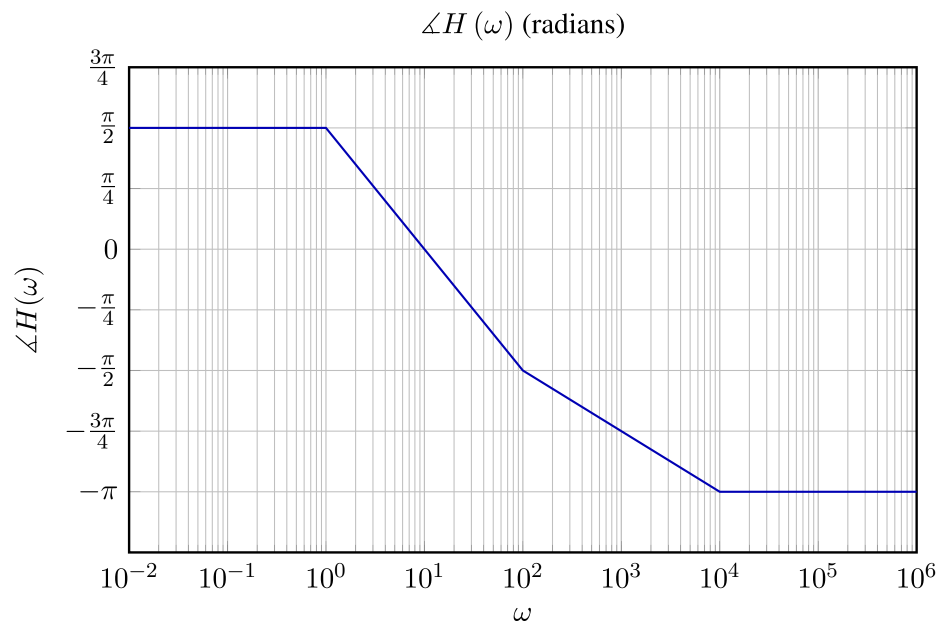
(c) **[6 points]** You decided to analyze the transfer function of a band-pass filter, and have generated the Bode plots in Figure 2a and Figure 2b for $H(\omega)$. **If your input voltage signal is**

$$v_{\text{in}}(t) = 10\cos\left(\omega_s t + \frac{\pi}{3}\right), \tag{9}$$

where $\omega_s = 10^4$, **what is** $v_{\text{out}}(t)$? *Show your work and explain your answers. You do not need to copy the figures below to your answer sheet, you may just tell us what you read from the Bode plots.*



(a) Magnitude Bode Plot for part (c).



(b) Phase Bode Plot for part (c).

Figure 2: Bode Plots for part (c).

Solution:

$$\boxed{v_{\text{out}}(t) = 0.1\cos\left(10^4 t + \frac{4\pi}{3}\right) \equiv 0.1\cos\left(10^4 t - \frac{2\pi}{3}\right)}$$

Given the Bode plots, we need to examine how the transfer function affects two quantities: the magnitude of the input voltage, and the phase of the input voltage. The Magnitude Bode Plot reveals that at $\omega = 10^4$, the value is 10^{-2}. The Phase Bode Plot reveals that at $\omega = 10^4$, the value is $-\pi$ radians. The general form of the output voltage is:

$$v_{\text{out}}(t) = |H(\omega)||v_{\text{in}}(t)| \cos\big(\omega t + \phi + \measuredangle H(\omega)\big) \tag{10}$$

where ϕ is the phase of the input voltage (here, $\frac{\pi}{3}$). Combining these results, we find:

$$v_{\text{out}}(t) = 0.01 \cdot 10 \cos\left(10^4 t + \frac{\pi}{3} - \pi\right) \tag{11}$$

$$= 0.1 \cos\left(10^4 t - \frac{2\pi}{3}\right) \tag{12}$$

4. Analyzing an LC-LC Band-Stop/Notch Filter

In this sub-part, you will partially analyze a circuit built entirely out of L, C components as shown in Figure 3. Assume that the circuit is operating at a frequency of $\omega = \omega_s$ (i.e. $v_{\text{in}}(t) = \cos(\omega_s t)$).

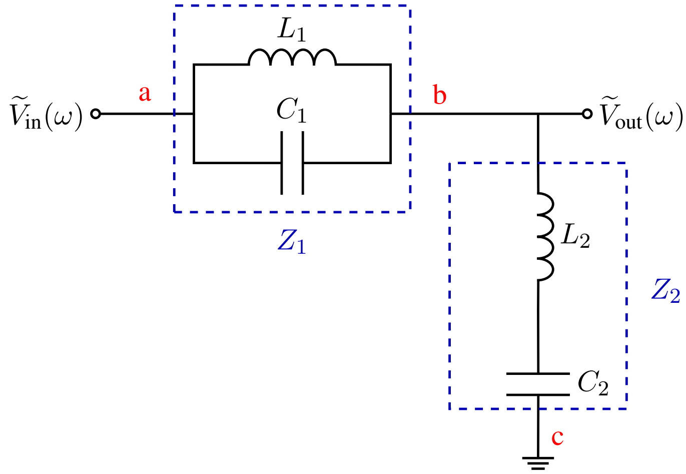


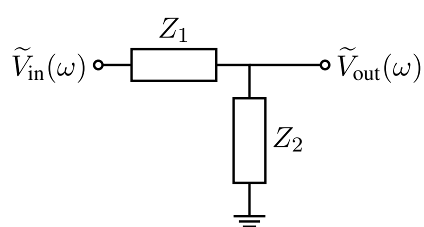
Figure 3: LC bandstop filter.

(a) **[5 points] Find $\widetilde{V}_{\text{out}}(\omega)$ in terms of Z_1, Z_2, $\widetilde{V}_{\text{in}}(\omega)$.** *You do not need to compute $\widetilde{V}_{\text{in}}(\omega)$ for this part. Show your work.*

Solution:

$\boxed{\widetilde{V}_{\text{out}}(\omega) = \frac{Z_2}{Z_1+Z_2}\widetilde{V}_{\text{in}}(\omega)}$

The equivalent circuit we get is as follows:



Since we are in the phasor domain, these impedances can be treated as resistors, and our output voltage phasor can be found as a function of the input voltage phasor by applying the voltage-divider equation. Doing so, we find that:

$$\widetilde{V}_{\text{out}}(\omega) = \frac{Z_2}{Z_1 + Z_2}\widetilde{V}_{\text{in}}(\omega) \tag{13}$$

(b) **[5 points] Find Z_1, the equivalent impedance between terminals a and b, in terms of L_1, C_1, and** ω_s. Leave your answer in the form $j\frac{M}{N}$, where M and N are real.
What is the impedance Z_1 at $\omega_s = \frac{1}{\sqrt{L_1 C_1}}$? *Show your work and justify your answers.*

Solution:

$Z_1 = \frac{j\omega_s L_1}{1-\omega_s^2 L_1 C_1}$. For $\omega_s = \frac{1}{\sqrt{L_1 C_1}}$, $Z_1 = \infty$

The impedances are in parallel. Using that $Z_L = j\omega_s L$ and $Z_C = \frac{1}{j\omega_s C}$ (at the frequency ω_s of the input phasor), we find:

$$\begin{aligned}
Z_1 &= Z_{L_1} \parallel Z_{C_1} \\
&= \frac{Z_{L_1} Z_{C_1}}{Z_{L_1} + Z_{C_1}} \\
&= \frac{j\omega_s L_1 \frac{1}{j\omega_s C_1}}{j\omega_s L_1 + \frac{1}{j\omega_s C_1}} \\
&= \frac{j\omega_s L_1}{(j\omega_s L_1 j\omega_s C_1) + 1} \\
&= \frac{j\omega_s L_1}{1 - \omega_s^2 L_1 C_1}
\end{aligned}$$

At $\omega_s = \frac{1}{\sqrt{L_1 C_1}}$:

$$\begin{aligned}
Z_1 &= \frac{j\omega_s L_1}{1 - \omega_s^2 L_1 C_1} \\
Z_1 &= \frac{j\omega_s L_1}{0} \\
Z_1 &= \infty
\end{aligned}$$

(c) **[5 points] Find Z_2, the equivalent impedance between terminals b and c, in terms of L_2, C_2, and** ω_s. Leave your answer in the form $j\frac{M}{N}$, where M and N are real.
What is the impedance Z_2 at $\omega_s = \frac{1}{\sqrt{L_2 C_2}}$? *Show your work and justify your answers.*

Solution:

$Z_2 = j\left(\omega_s L_2 - \frac{1}{\omega_s C_2}\right)$. For $\omega_s = \frac{1}{\sqrt{L_2 C_2}}$, $Z_2 = 0$.

Simplifying the series LC combination, we find (since $\frac{1}{j} = -j$):

$$\begin{aligned}
Z_2 &= Z_{L_2} + Z_{C_2} \\
&= j\omega_s L_2 + \frac{1}{j\omega_s C_2} \\
&= j\left(\omega_s L_2 - \frac{1}{\omega_s C_2}\right)
\end{aligned}$$

At $\omega_s = \frac{1}{\sqrt{L_2 C_2}}$:

$$Z_2 = j\left(\frac{L_2}{\sqrt{L_2 C_2}} - \frac{\sqrt{L_2 C_2}}{C_2}\right)$$

$$
\begin{aligned}
&= j\left(\sqrt{\frac{L_2}{C_2}} - \sqrt{\frac{L_2}{C_2}}\right) \\
&= 0
\end{aligned}
$$

5. Hey Circuit, are you a Low-Pass Filter?

You have a mystery black box and you believe it contains an RC low-pass filter. You want to use the tools of converting a model from continuous-time to discrete-time and System ID to test your guess.

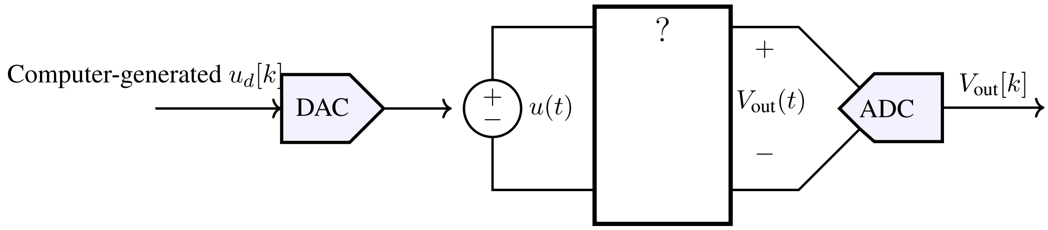


Figure 4: A schematic to show how our computer generated signals will interface through the DAC and the ADC with the mystery box.

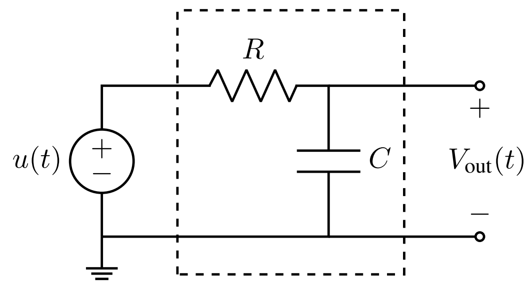


Figure 5: RC circuit that you suspect is inside the mystery (?) box.

(a) **[4 points]** You know from lecture that the continuous-time equation

$$\frac{dx(t)}{dt} = \lambda x(t) + bu(t), \tag{14}$$

can be converted to a discrete-time equation given by

$$x_d[k+1] = e^{\lambda\Delta} x_d[k] + b\left(\frac{e^{\lambda\Delta} - 1}{\lambda}\right) u_d[k], \tag{15}$$

where $x_d[k] = x(k\Delta)$ and $u_d[k] = u(k\Delta)$, for some constant Δ.
Assume that eq. (14) references the RC circuit in Figure 5. For a low-pass RC filter with input $u(t)$ you know that the following differential equation holds:

$$\frac{dV_{\text{out}}(t)}{dt} = -\frac{1}{RC} V_{\text{out}}(t) + \frac{1}{RC} u(t). \tag{16}$$

Convert this continuous system to a discrete-time difference equation for $V_{\text{out}}[k]$ in the form of

$$V_{\text{out}}[k+1] = \lambda_d V_{\text{out}}[k] + b_d u_d[k], \tag{17}$$

and write λ_d and b_d in terms of R, C, and Δ. *Show your work.*

Solution: Comparing eq. (16) to eq. (14) we infer that $\lambda = -\frac{1}{RC}$ and $b = \frac{1}{RC}$. Using the continuous to discrete time conversion:

$$V_{\text{out}}[k+1] = e^{-\frac{\Delta}{RC}} V_{\text{out}}[k] + \left(1 - e^{-\frac{\Delta}{RC}}\right) u_d[k],$$

$$\lambda_d = e^{-\frac{\Delta}{RC}} \qquad b_d = \left(1 - e^{-\frac{\Delta}{RC}}\right).$$

(b) **[5 points]** Now ignoring the physics of the model, you decide to use a data-centric approach to find λ_d and b_d in your model. In order to do so, you apply a sequence of inputs for 4 timesteps $u_d[0], u_d[1], u_d[2], u_d[3]$ and observe $V_{\text{out}}[0], V_{\text{out}}[1], V_{\text{out}}[2], V_{\text{out}}[3]$, and $V_{\text{out}}[4]$. You decide to use least squares of the form $D\vec{p} \approx \vec{y}$, with $\vec{p} = \begin{bmatrix} \lambda_d \\ b_d \end{bmatrix}$. **Write the matrix D and vector $\vec{y}$.** *Write your answers in the provided box.*

Solution:

$$D = \begin{bmatrix} V_{\text{out}}[0] & u_d[0] \\ V_{\text{out}}[1] & u_d[1] \\ V_{\text{out}}[2] & u_d[2] \\ V_{\text{out}}[3] & u_d[3] \end{bmatrix}, \qquad y = \begin{bmatrix} V_{\text{out}}[1] \\ V_{\text{out}}[2] \\ V_{\text{out}}[3] \\ V_{\text{out}}[4] \end{bmatrix}. \tag{18}$$

6. A Spring System

The tools we have learned in this class are not limited to circuits. In this problem we will examine the following spring-mass system. This system can be modeled with eq. (19), where $x(t)$ is the position of the mass at time t, m is the constant mass of the block, and $u(t)$ is the force input to the system at time t.

$$\frac{d^2x(t)}{dt^2} = -\frac{k_1}{m}x(t) - \frac{k_2}{m}\frac{dx(t)}{dt} + \frac{1}{m}u(t). \tag{19}$$

(a) **[4 points] Rewrite eq. (19) as a system of differential equations in the matrix form.** Let the state variables be $\vec{y} = \begin{bmatrix} x(t) \\ \frac{dx(t)}{dt} \end{bmatrix}$. *Show your work.*

Solution: Since we have defined our state as $\vec{y}(t) = \begin{bmatrix} x(t) \\ \frac{dx(t)}{dt} \end{bmatrix}$, to write this in matrix form we need equations for $\frac{dx(t)}{dt}$ and $\frac{d^2x(t)}{dt^2}$ in terms of $x(t)$, $\frac{dx(t)}{dt}$, and $u(t)$.

$$\frac{dx(t)}{dt} = 0 \cdot x(t) + 1 \cdot \frac{dx(t)}{dt} + 0 \cdot u(t) \tag{20}$$

$$\frac{d^2x(t)}{dt^2} = -\frac{k_1}{m} \cdot x(t) - \frac{k_2}{m} \cdot \frac{dx(t)}{dt} + \frac{1}{m} \cdot u(t) \tag{21}$$

Equation (20) comes from setting $\frac{dx(t)}{dt}$ equal to itself. Equation (21) is identical to equation (19). Putting these two equations together in the form $\frac{d\vec{y}(t)}{dt} = A\vec{y}(t) + bu(t)$ we get the following system of differential equations in matrix form:

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{d^2x(t)}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m} & -\frac{k_2}{m} \end{bmatrix} \begin{bmatrix} x(t) \\ \frac{dx(t)}{dt} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t).$$

(b) **[10 points]** Regardless of your answer to the previous question, assume that you end up with the following system:

$$\frac{d\vec{y}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \vec{y}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t). \tag{22}$$

The matrix $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ can be diagonalized as $A = V\Lambda V^{-1}$ where

$$V = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}, \tag{23}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \tag{24}$$

$$V^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}. \tag{25}$$

If the input is fixed to a constant $u(t) = u_0 \in \mathbb{R}$ for all t, **find the solution to the system of differential equations in eq. (22).** Use $\vec{y}(0) = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ as the initial condition. *Show your work and justify your answers.*

Solution: We can transform the original system into the eigenbasis by left-multiplying eq. (22) by V^{-1} and simplifying:

$$\frac{d\vec{y}}{dt}(t) = A\vec{y}(t) + \vec{b}u(t) \tag{26}$$

$$V^{-1}\frac{d\vec{y}}{dt}(t) = V^{-1}A\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{27}$$

$$\frac{d\widetilde{\vec{y}}}{dt}(t) = V^{-1}(V\Lambda V^{-1})\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{28}$$

$$= \Lambda\widetilde{\vec{y}}(t) + V^{-1}\vec{b}u(t) \tag{29}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\widetilde{\vec{y}} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_0. \tag{30}$$

This gives us the following decoupled system of differential equations:

$$\frac{d\widetilde{y}_1}{dt}(t) = -\widetilde{y}_1(t) + u_0 \tag{31}$$

$$\frac{d\widetilde{y}_2}{dt}(t) = -3\widetilde{y}_2(t) - u_0. \tag{32}$$

Solving eq. (31) and eq. (32) gives:

$$\widetilde{y}_1(t) = \left(\widetilde{y}_1(0) - u_0\right)e^{-t} + u_0 \tag{33}$$

$$\widetilde{y}_2(t) = \left(\widetilde{y}_2(0) + \frac{u_0}{3}\right)e^{-3t} - \frac{u_0}{3}. \tag{34}$$

The initial conditions of $\widetilde{\vec{y}}$ are:

$$\widetilde{\vec{y}}(0) = V^{-1}\vec{y}(0) \tag{35}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3\alpha}{2} \\ \frac{-\alpha}{2} \end{bmatrix}. \tag{36}$$

Plugging in these initial conditions into eq. (33) and eq. (34) gives the solutions in the eigenbasis as:

$$\widetilde{y}_1(t) = \left(\frac{3\alpha}{2} - u_0\right)e^{-t} + u_0, \tag{37}$$

$$\widetilde{y}_2(t) = \left(-\frac{\alpha}{2} + \frac{u_0}{3}\right)e^{-3t} - \frac{u_0}{3}. \tag{38}$$

To transform these solutions back to our original basis, recall $\vec{y} = V\widetilde{\vec{y}}$:

$$\vec{y} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} \left(\frac{3\alpha}{2} - u_0\right)e^{-t} + u_0 \\ \left(-\frac{\alpha}{2} + \frac{u_0}{3}\right)e^{-3t} - \frac{u_0}{3} \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} \left(\frac{3\alpha}{2} - u_0\right) e^{-t} + \left(-\frac{\alpha}{2} + \frac{u_0}{3}\right) e^{-3t} + \frac{2u_0}{3} \\ -\left(\frac{3\alpha}{2} - u_0\right) e^{-t} - 3\left(-\frac{\alpha}{2} + \frac{u_0}{3}\right) e^{-3t} \end{bmatrix}. \tag{40}$$

Thus we get our final solutions in terms of the original variables:

$$x(t) = \left(\frac{3\alpha}{2} - u_0\right) e^{-t} + \left(-\frac{\alpha}{2} + \frac{u_0}{3}\right) e^{-3t} + \frac{2u_0}{3}, \tag{41}$$

$$\frac{dx(t)}{dt} = -\left(\frac{3\alpha}{2} - u_0\right) e^{-t} - 3\left(-\frac{\alpha}{2} + \frac{u_0}{3}\right) e^{-3t}. \tag{42}$$

(c) **[5 points]** Suppose you apply a piece-wise constant input to this system, such that $u(t)$ is constant over intervals of Δ:

$$u(t) = u(i\Delta) = u_d[i] \text{ for } t \in [i\Delta, (i+1)\Delta), \tag{43}$$

and suppose your solution to the differential equation for the spring-mass system for $t \in (i\Delta, (i+1)\Delta]$ is

$$y_1(t) = (y_1(i\Delta) - u_d[i])e^{-(t-i\Delta)} + 2 \cdot y_2(i\Delta)e^{-2(t-i\Delta)} + u_d[i], \tag{44}$$

$$y_2(t) = -(y_1(i\Delta) - u_d[i])e^{-(t-i\Delta)} - 4 \cdot y_2(i\Delta)e^{-2(t-i\Delta)}, \tag{45}$$

Given the initial conditions $\vec{y}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and input $u_d[0] = 4$, find $\vec{y}(\Delta)$. Assume $e^{-\Delta} = 0.5$. *Show your work and justify your answers.*

Solution: To find $\vec{y}(\Delta)$ we should just plug in Δ into the expression in eq. (44) and eq. (45):

$$\begin{aligned} y_1(\Delta) &= (y_1(0) - u_d[0]) \cdot e^{-\Delta} + 2y_2(0) \cdot e^{-2\Delta} + u_d[0] \\ &= (4-4) \cdot e^{-\Delta} + 2 \cdot 0 \cdot e^{-2\Delta} + 4 = 4. \end{aligned}$$

$$\begin{aligned} y_2(\Delta) &= -(y_1(0) - u_d[0]) \cdot e^{-\Delta} - 4y_2(0) \cdot e^{-2\Delta} \\ &= -(4-4) \cdot e^{-\Delta} - 4 \cdot 0 \cdot e^{-2\Delta} = 0. \end{aligned}$$

(d) **[4 points]** We decide to examine stability of this system in discrete time, so we fix Δ and derive the following discretized system:

$$\vec{y}_d[k+1] = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \vec{y}_d[k] + \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} u_d[k]. \tag{46}$$

Identify if this open-loop system is stable. *Show your work and justify your answers.*

Solution: Since this is a discrete system, in order to test if the open-loop system is stable, we need to compute the eigenvalues of the state transition matrix $\left(\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}\right)$ and check if all their magnitudes are less than one.

We follow the familiar procedure to form the characteristic polynomial:

$$\det\left(\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \tag{47}$$

$$\det\left(\begin{bmatrix} \frac{3}{4} - \lambda & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} - \lambda \end{bmatrix}\right) = 0 \tag{48}$$

$$\left(\frac{3}{4} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \left(-\frac{1}{4} \cdot \frac{1}{4}\right) = 0 \tag{49}$$

$$\lambda^2 - \lambda + \frac{3}{16} + \frac{1}{16} = 0 \tag{50}$$

$$\lambda^2 - \lambda + \frac{1}{4} = 0 \tag{51}$$

$$\left(\lambda - \frac{1}{2}\right)^2 = 0 \tag{52}$$

$$\implies \lambda_1 = \lambda_2 = \frac{1}{2} \tag{53}$$

From the above analysis, we see that the eigenvalues of the state transition matrix are $\lambda_1 = \lambda_2 = 0.5$. Since these both have magnitude less than 1, the system is stable.

(e) **[4 points]** If we put the system defined in eq. (46) in feedback, setting $u_d[k] = \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{y}_d[k]$, **is the resulting closed-loop system stable?** *Show your work and justify your answers.*

Solution: Here, we have closed-loop feedback since the input is expressed as a scalar that depends on the current value of the state vector. Making the given substitution, our system becomes

$$\vec{y}_d[k+1] = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \vec{y}_d[k] + \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{y}_d[k] \tag{54}$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \vec{y}_d[k] + \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \vec{y}_d[k] \tag{55}$$

$$= \left(\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \right) \vec{y}_d[k] \tag{56}$$

$$= \begin{bmatrix} \frac{6}{4} & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \vec{y}_d[k] \tag{57}$$

Now that we have a single matrix, we can compute it's eigenvalues:

$$\det \left(\begin{bmatrix} \frac{3}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0 \tag{58}$$

$$\det \left(\begin{bmatrix} \frac{3}{2} - \lambda & 1 \\ 0 & \frac{1}{2} - \lambda \end{bmatrix} \right) = 0 \tag{59}$$

$$\left(\frac{3}{2} - \lambda \right) \left(\frac{1}{2} - \lambda \right) = 0 \tag{60}$$

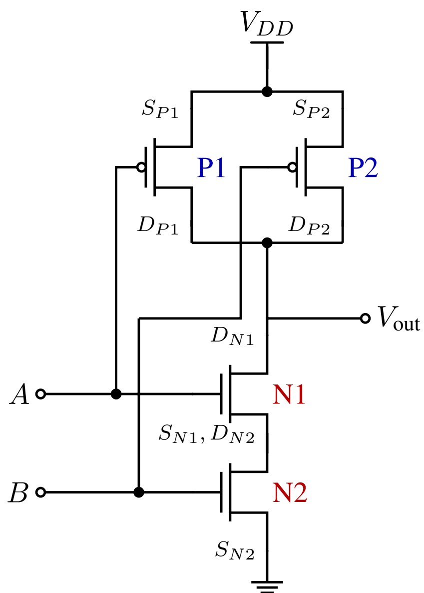
$$\implies \lambda_1 = \frac{3}{2}, \lambda_2 = \frac{1}{2} \tag{61}$$

We computed that the eigenvalues that govern the dynamics of this closed-loop system are $\lambda_1 = 1.5$ and $\lambda_2 = 0.5$. Since $|\lambda_1| > 1$, this discrete-time system is unstable.

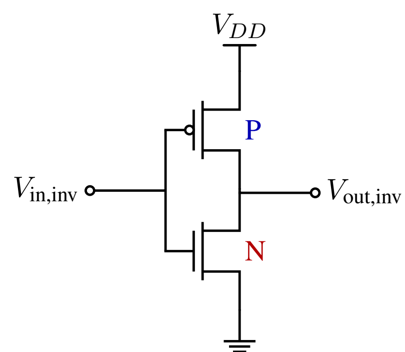
7. Transistor Switch Model

In this problem, we will analyze the behavior of a NAND gate driving an inverter. Figure 6a shows the transistor model of a NAND gate and Figure 6b shows the transistor model of an inverter.

In this question assume that V_{DD} is greater than both the NMOS threshold $V_{\text{th,n}}$ and PMOS threshold $|V_{\text{th,p}}|$.



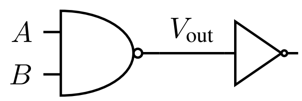
(a) NAND schematic with transistors.



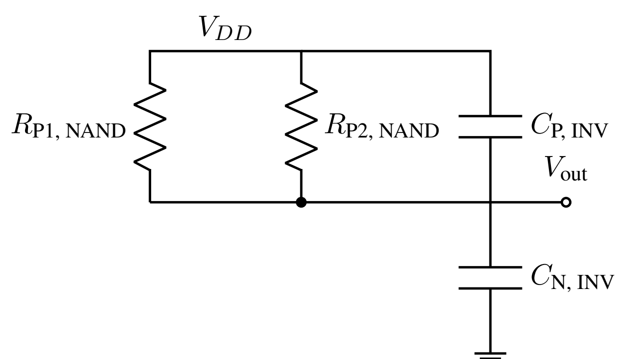
(b) Inverter schematic with transistors.

Figure 6: Transistor schematics

(a) **[5 points]** A diagram of a NAND gate driving an inverter is shown in Figure 7a. Consider the case where $A = V_{DD}$ and $B = V_{DD}$ for a long time before $t = 0$. Then at $t = 0$, we switch A and B to $0\,\text{V}$. The equivalent simplified circuit after this transition is shown in Figure 7b. **Find** V_{out} **at time** $t = 0$. *Write your answers in the provided box.*



(a) NAND driving an Inverter



(b) A Simplified Equivalent Circuit for when $A = B = 0\text{V}$

Figure 7: Schematic and model of a NAND gate driving an inverter

Solution: $V_{\text{out}}(0) = 0$. Since $A = B = V_{DD}$ for $t < 0$, both NMOS transistors have been switched on due to $V_{GSn} = V_{DD} > V_{\text{th,n}}$ until right before $t = 0$. Similarly, both PMOS transistors are switched off because $|V_{GSp}| = 0 < |V_{\text{th,p}}|$. Therefore, $C_{\text{N, INV}}$ is fully discharged to 0V and $C_{\text{P, INV}}$ is fully charged to V_{DD} right before $t = 0$. When the transition happens the charge on capacitors cannot jump instantaneously. So, at $t = 0$ the voltage at output will remain 0V.

(b) **[8 points] Write the differential equation for solving $V_{\text{out}}(t)$ for $t \geq 0$ in the circuit shown in Figure 7b**. Specifically, **find coefficients λ and b in the following symbolic differential equation:**

$$\frac{dV_{\text{out}}(t)}{dt} = \lambda V_{\text{out}}(t) + bV_{DD}, \tag{62}$$

as a function of $R_P, C_{\text{P, INV}}, C_{\text{N, INV}}$, and V_{DD}. Assume that $R_{\text{P1,NAND}} = R_{\text{P2, NAND}} = R_P$. *Show your work and justify your answers.*

Solution: The first thing to note is that the two resistors are in parallel, so we can lump them into one resistor with half the value. We can then start by writing a KCL at the output node:

$$C_{\text{N, INV}}\frac{dV_{\text{out}}(t)}{dt} + C_{\text{P, INV}}\frac{d(V_{\text{out}}(t) - V_{DD})}{dt} + \frac{2}{R_P}(V_{\text{out}}(t) - V_{DD}) = 0,$$

$$\frac{dV_{\text{out}}(t)}{dt} = -\frac{2}{R_P(C_{\text{P, INV}} + C_{\text{N, INV}})}V_{\text{out}}(t) + \frac{2}{R_P(C_{\text{P, INV}} + C_{\text{N, INV}})}V_{DD}.$$

Therefore, $\lambda = -\frac{2}{R_P(C_{\text{P, INV}}+C_{\text{N, INV}})}$ and $b = \frac{2}{R_P(C_{\text{P, INV}}+C_{\text{N, INV}})}$.

(c) **[8 points] Solve $V_{\text{out}}(t)$ in the differential equation (62) and the initial condition $V_{\text{out}}(0)$.** You should leave your answer in terms of λ, b, V_{DD}, and $V_{\text{out}}(0)$. *Show your work and justify your answers.*

Solution:

Approach 1: We know from homework that for a differential equation of type $\frac{dv(t)}{dt} = \lambda v(t) + bu(t)$ and initial condition $v(t_0)$, the answer is $v(t) = v(t_0)e^{\lambda t} + b\int_{t_0}^{t} u(\tau)e^{\lambda(t-\tau)}d\tau$. We will apply the same principle here as well:

$$\begin{aligned} V_{\text{out}}(t) &= V_{\text{out}}(0)e^{\lambda t} + b\int_0^t V_{DD}e^{\lambda(t-\tau)}d\tau & (63)\\ &= V_{\text{out}}(0)e^{\lambda t} + bV_{DD}e^{\lambda t}\int_0^t e^{-\lambda\tau}d\tau & (64)\\ &= V_{\text{out}}(0)e^{\lambda t} + bV_{DD}e^{\lambda t}\cdot\frac{e^{-\lambda t}-1}{-\lambda} & (65)\\ &= V_{\text{out}}(0)e^{\lambda t} + \frac{b}{\lambda}V_{DD}\left(e^{\lambda t}-1\right). & (66)\end{aligned}$$

Approach 2: We can use variable substitution to arrive at a homogeneous differential equation and then solve it. Define $\widetilde{V}_{\text{out}}(t) = V_{\text{out}}(t) + \frac{b}{\lambda}V_{DD}$.

$$\begin{aligned} \frac{d(V_{\text{out}}(t) + \frac{b}{\lambda}V_{DD})}{dt} &= \lambda\left(V_{\text{out}}(t) + \frac{b}{\lambda}V_{DD}\right), & (67)\\ \frac{d\widetilde{V}_{\text{out}}(t)}{dt} &= \lambda\widetilde{V}_{\text{out}}(t), & (68)\\ \widetilde{V}_{\text{out}}(t) &= \widetilde{V}_{\text{out}}(0)e^{\lambda t}. & (69)\end{aligned}$$

We also know that $\widetilde{V}_{\text{out}}(0) = V_{\text{out}}(0) + \frac{b}{\lambda} V_{DD}$. Substituting V_{out} back will give us the answer.

$$V_{\text{out}}(t) + \frac{b}{\lambda} V_{DD} = \left(V_{\text{out}}(0) + \frac{b}{\lambda} V_{DD} \right) e^{\lambda t} \tag{70}$$

$$= V_{\text{out}}(0) e^{\lambda t} + \frac{b}{\lambda} V_{DD} \left(e^{\lambda t} - 1 \right). \tag{71}$$

(d) **[8 points]** Now consider the case where $A = 0\text{V}$ and $B = 0\text{V}$ for a long time before $t = 0$ in Figure 7a. At $t = 0$ we switch A and B to V_{DD}. **Write down the state (ON/OFF) of transistors P1, P2, N1, and N2 in the NAND gate. Draw the equivalent simplified circuit for this transition that will help us with writing the differential equation of** $V_{\text{out}}(t)$. *Write your answers in the provided box.*

Hint: You may find the NAND resistor-switch model in Figure 8 helpful. Don't forget to include the inverter's capacitors, $C_{N,\,INV}$ and $C_{P,\,INV}$, which are loading the NAND gate.

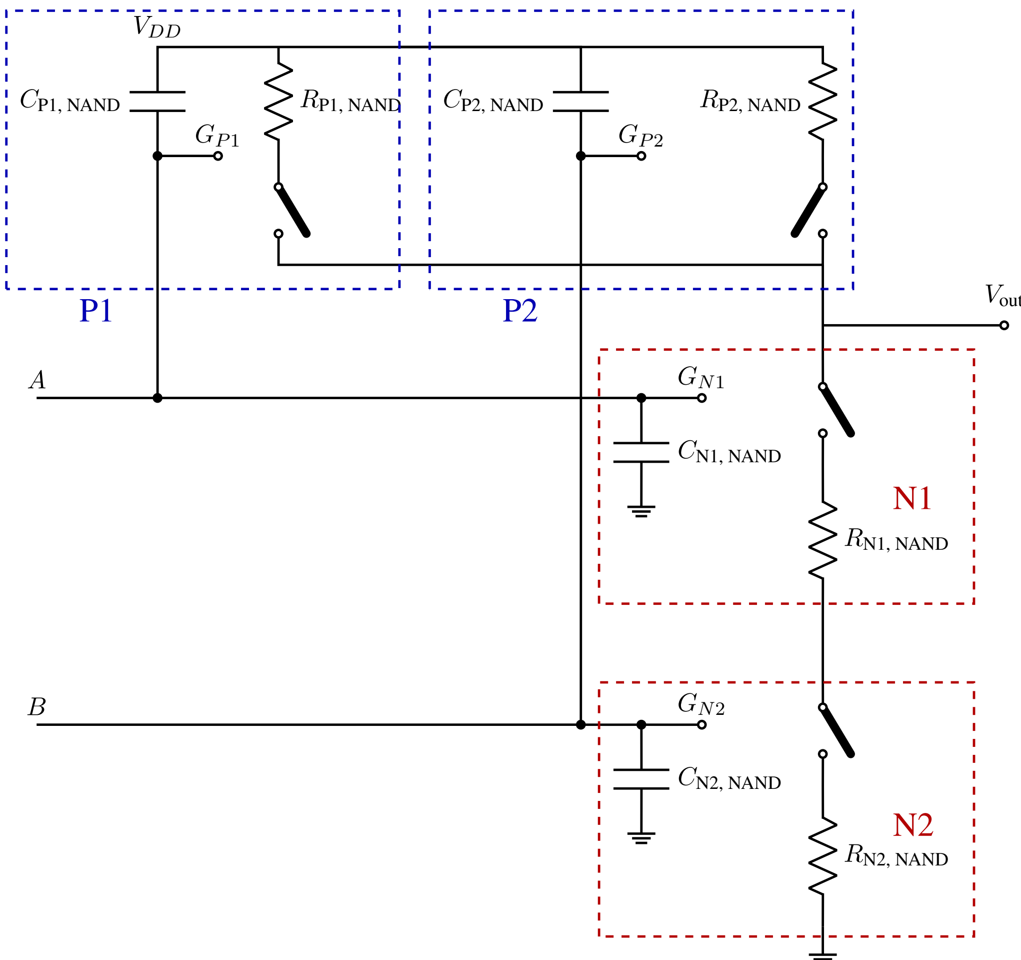


Figure 8: NAND Model: Capacitances

Solution:

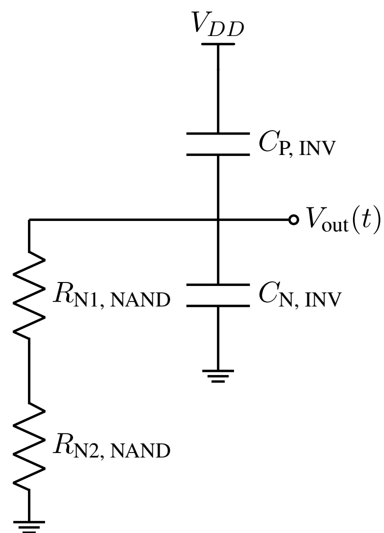
Both of our inputs are high (V_{DD}). Thus $|V_{\text{GSp}}| = 0V \implies$ both PMOS transistors P1 and P2 are OFF. At the same time $V_{\text{GSn}} = V_{\text{DD}} \implies$ both NMOS transistors N1 and N2 are ON.

Since the PMOS transistors are both off, their switches in the equivalent circuit are open. As a result their resistors $R_{\text{P1, NAND}}$ and $R_{\text{P2, NAND}}$ are floating and don't need to be included.

The NMOS transistors, on the other hand, are both on, so in the simplified equivalent circuit, their resistors $R_{\text{N1, NAND}}$ and $R_{\text{N2, NAND}}$ connect the output to ground.

In addition, V_{out} drives an inverter, which means that V_{out} is connected to two gate capacitances: $C_{\text{P, INV}}$ to V_{DD} and $C_{\text{N, INV}}$ to ground.

Thus we end up with the simplified circuit below.



8. Loud Neighbors

The neighbors keep throwing loud parties and Divija is having trouble sleeping despite her ear plugs. She decides to build a device to reduce the noise, and needs your help designing the filters.

(a) **[4 points]** Divija decides to build a band-stop filter by combining a low-pass and a high-pass filter. To start off, consider the skeleton circuit in Figure 9. **What is V_{out} in terms of u_1, u_2, and R?**

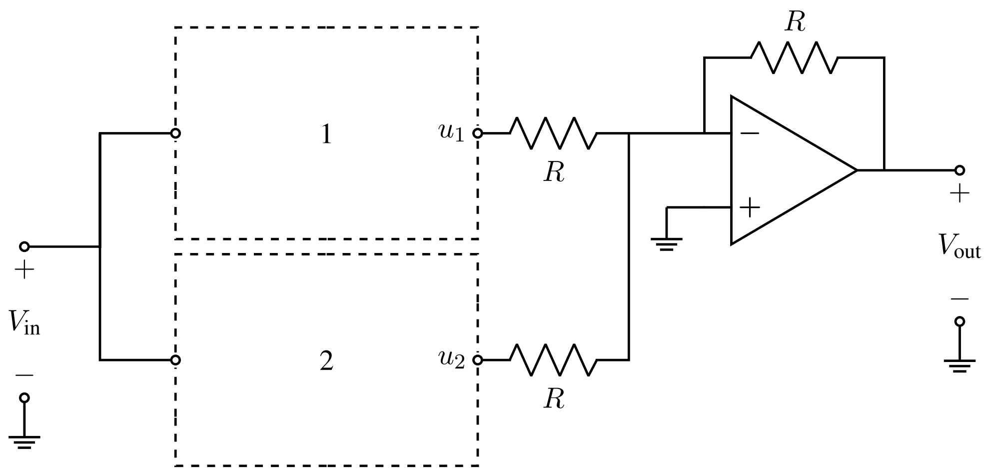


Figure 9: Skeleton Circuit for part (a).

Solution: Since this ideal op-amp is in negative feedback, $v_- = v_+$. Doing KCL at v_- we get:

$$\frac{u_1}{R} + \frac{u_2}{R} = -\frac{V_{\text{out}}}{R}.$$

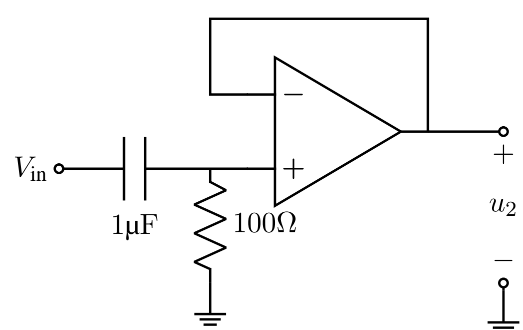
Solving for V_{out} we get:

$$V_{\text{out}} = -(u_1 + u_2).$$

(b) **[6 points] Design a *high-pass* filter for Box 2.** This should be a circuit with cutoff frequency $\omega_c = 10^4 \text{rad/s}$ that can drive an arbitrary load. You may use one resistor, one op-amp, and one $1\mu\text{F} = 10^{-6}\text{F}$ capacitor. **Choose the value of the resistor to get the correct cut-off frequency ω_c.** *Show your work and justify your answers.*

[Question Continues on Next Page]

Solution:



(c) **[6 points]** Divija designed the **low-pass** filter shown in Figure 10 for a cut-off frequency of $\omega_c = 10^2 \mathrm{rad/s}$ to be used in Box 1. To verify that the circuit she built matched the circuit she designed, she decided to test the circuit in isolation by applying an input V_{in} and measuring the filter's output V_{out}. The input output behavior of the circuit she built is shown in Figure 11.

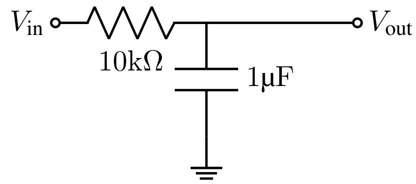


Figure 10: The filter Divija intended to build.

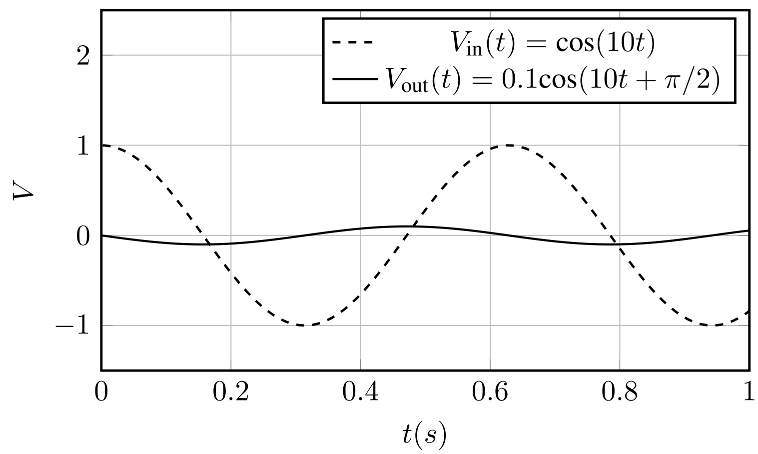


Figure 11: Input-Output behavior of the filter Divija built.

What is the most likely cause for this behavior? *Show your work and justify your answers.*

i. The resistors and capacitors were swapped.
ii. She used an inductor instead of the resistor.
iii. She used a $10\mu\text{F}$ capacitor and a resistor of $1\text{k}\Omega$.

Solution: (i)
Note that the V_{in} in the graph has a low frequency (time is in units of seconds).

i. Correct: This would result in a high pass filter, which would attenuate the low-frequency V_{in} as shown.
ii. Incorrect: At low frequencies, the swapping the resistor with an inductor would still preserve low frequencies.
iii. Incorrect: Multiplying the capacitor and dividing the resistor values by 10 would not change the transfer function.

[Question Continues on Next Page]

(d) **[8 points]** After looking through the available components Divija realizes that she doesn't have enough capacitors and decides to build the filter with inductors instead. Assume she builds the overall circuit in Figure 12. **Find the transfer functions $H_1(\omega) = \frac{\widetilde{u}_1}{\widetilde{V}_{\text{in}}}$ and $H_2(\omega) = \frac{\widetilde{u}_2}{\widetilde{V}_{\text{in}}}$ in terms of L_1, L_2, R_1, R_2, and R_s.** *Show your work and justify your answers.*

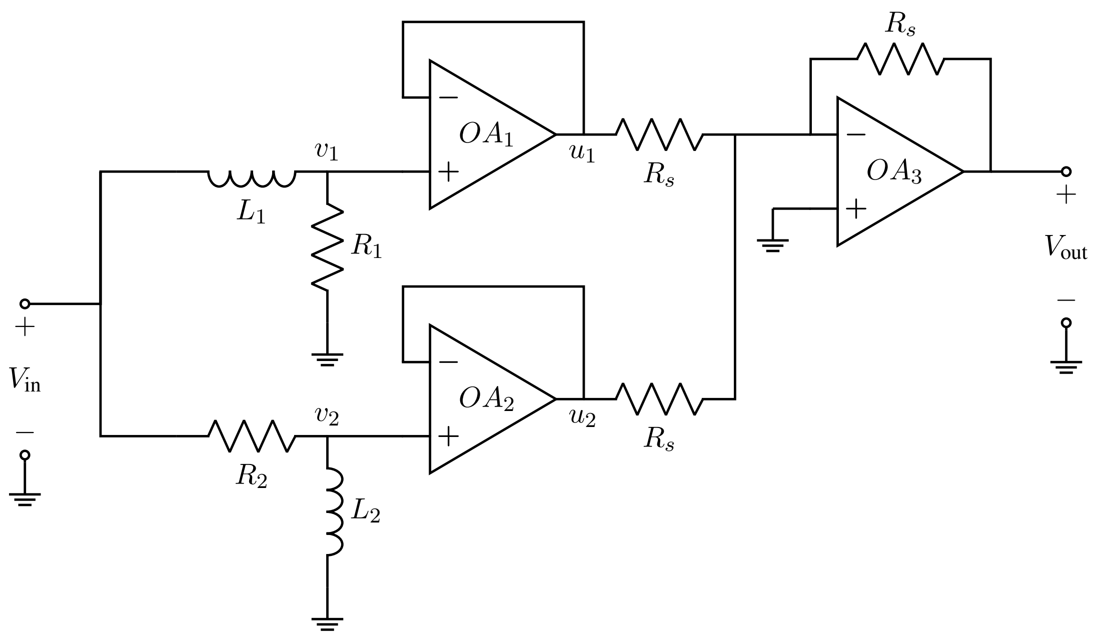


Figure 12: Overall Circuit

Solution:

Notice that OA_1 is in the buffer configuration, so $\widetilde{u}_1 = \widetilde{v}_1$. We can use the voltage divider formula to solve for $\widetilde{v}_1$:

$$\widetilde{v}_1 = \frac{Z_{\text{bottom}}}{Z_{\text{total}}}\widetilde{V}_{\text{in}} = \frac{\widetilde{V}_{\text{in}} R_1}{R_1 + j\omega L_1}. \tag{72}$$

Thus

$$H_1(\omega) = \frac{R_1}{R_1 + j\omega L_1} = \frac{1}{1 + j\omega L_1/R_1}.$$

Applying the same analysis to u_2 we see that OA_2 is also in the buffer configuration. Thus $\widetilde{u}_2 = \widetilde{v}_2$. Again, we can use the voltage divider equation to solve for $\widetilde{v}_2$:

$$\widetilde{v}_2 = \frac{Z_{\text{bottom}}}{Z_{\text{total}}}\widetilde{V}_{\text{in}} = \frac{\widetilde{V}_{\text{in}} j\omega L_2}{R_1 + j\omega L_2}. \tag{73}$$

Thus

$$H_2\omega) = \frac{j\omega L_2}{R_2 + j\omega L_2} = \frac{j\omega L_2/R_2}{1 + j\omega L_2/R_2}.$$

(e) **[6 points]** Assume the overall transfer function of the final circuit in Figure 12, $H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$, is

$$H(\omega) = \left(\frac{1}{1 + j\omega/\omega_{c1}} + \frac{j\omega/\omega_{c2}}{1 + j\omega/\omega_{c2}}\right), \tag{74}$$

where $\omega_{c2} = 100\omega_{c1}$. **Qualitatively describe the magnitude of the transfer function $|H(\omega)|$ in three regions: frequencies below ω_{c1}, frequencies between ω_{c1} and ω_{c2}, and frequencies above ω_{c2}. Explain what the filter is doing qualitatively** (for example, a low-pass filter passes low frequencies but does not pass high frequencies). *Show your work and justify your answers.*

Solution: We can qualitatively analyze the behavior of the transfer function by taking the limits when $\omega = 0$ and $\omega \to \infty$. We can see that $|H(0)| = 1$ and $|H(\infty)| = 1$. If we compute $H(\omega)$ for $\omega \in [\omega_{c1}, \omega_{c2}]$ we can see that $|H(\omega)| < 1$. For example at $\omega_0 = 10\omega_{c_1}$, $|H(\omega_0)| = \frac{\sqrt{2}}{101} \ll 1$. Therefore, this filter does not pass the frequencies between ω_{c1} and ω_{c2}, forming a band-stop filter.

