EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet

Discussion 8B

The following notes are useful for this discussion: Note 12.

1. Gram-Schmidt Algorithm

Let's apply Gram-Schmidt orthonormalization to a list of three linearly independent vectors $[\vec{s}_1, \vec{s}_2, \vec{s}_3]$.

(a) **Find unit vector $\vec{q}_1$ such that $\mathrm{Span}(\{\vec{q}_1\}) = \mathrm{Span}(\{\vec{s}_1\})$.**

Solution: Since $\mathrm{Span}(\{\vec{s}_1\})$ is a one dimensional vector space, the unit vector that spans the same vector space would just be the normalized vector which points in the same direction as $\vec{s}_1$. Therefore

$$\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|}. \tag{1}$$

(b) Given $\vec{q}_1$ from the previous step, **find unit vector $\vec{q}_2$ such that $\mathrm{Span}(\{\vec{q}_1, \vec{q}_2\}) = \mathrm{Span}(\{\vec{s}_1, \vec{s}_2\})$ and $\vec{q}_2$ is orthogonal to $\vec{q}_1$.**

Solution: We want to find the projection of $\vec{s}_2$ onto $\vec{q}_1$. The last time we worked with this type of projections is when we discussed least squares. We will try to work in that framework. A detailed discussion of least squares is in 16A Note 23.

Specifically, least squares solves the problem $A\vec{x} \approx \vec{b}$ by finding the closest point $A\vec{x}$ to $\vec{b}$ among all vectors $\vec{x}$. Since $A\vec{x}$ ranges over all vectors in $\mathrm{Col}(A)$, this is equivalent to finding the closest $\vec{b}_*$ in $\mathrm{Col}(A)$ to $\vec{b}$. We know that

$$\vec{x} = \left(A^\top A\right)^{-1} A^\top \vec{b} \implies \vec{b}_* = A\left(A^\top A\right)^{-1} A^\top \vec{b}. \tag{2}$$

Thus, to project a vector into the column space of A, we multiply it by the matrix $A\left(A^\top A\right)^{-1} A^\top$.

In particular here, we want to project $\vec{s}_2$ onto $\vec{q}_1$. One matrix A such that $\mathrm{Span}(\vec{q}_1) = \mathrm{Col}(A)$ is just $A = \vec{q}_1$. Since $\vec{q}_1$ is normalized:

$$A\left(A^\top A\right)^{-1} A^\top = \vec{q}_1 (\underbrace{\vec{q}_1^\top \vec{q}_1}_{1})^{-1} \vec{q}_1^\top = \vec{q}_1 \vec{q}_1^\top \implies A\left(A^\top A\right)^{-1} A^\top \vec{s}_2 = \vec{q}_1 \vec{q}_1^\top \vec{s}_2 = \left(\vec{q}_1^\top \vec{s}_2\right)\vec{q}_1. \tag{3}$$

Subtracting out this projection gets us *only the component orthogonal to $\vec{q}_1$*:

$$\vec{z}_2 = \vec{s}_2 - \left(\vec{q}_1^\top \vec{s}_2\right)\vec{q}_1. \tag{4}$$

Note that this projection formula *only works if $\vec{q}_1$ is normalized*. Now, normalizing to get $\vec{q}_2$, we have $\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|}$.

(c) **Suppose we want to show that $\mathrm{Span}(\{\vec{q}_1, \vec{q}_2\}) = \mathrm{Span}(\{\vec{s}_1, \vec{s}_2\})$. What does this mean mathematically?** *Hint: you cannot use the word span, but must capture the same concept in your translation of the statement we want to show.*

Solution: To translate the statement, we need to first understand what kind of object is a span. A span is an entire subspace — an infinite set of vectors. We are trying to assert that two spans are the same, which means that we want to assert that two infinite sets are in fact different names for the same infinite set. How do we do this? We need to say that any vector in the first set is in the second set. This tells us that the first set is a subset of the second set. So we also need to say the reverse direction: that any vector in the second set is in the first set. If two sets are both subsets of each other, then they have to be different names for the same set.

Understanding the types of the objects in question (in this case, subspaces) moves us closer to translating the statement. But it doesn't get us all the way there. We need a way to point to a specific element of the first set, and then we need a way of arguing that it is indeed in the second set. For this, presumably, we will need to use the actual definition of span.

At this point, we know what to do. We say $\forall \alpha, \beta, \exists \gamma, \delta$ so that $\alpha\vec{q}_1 + \beta\vec{q}_2 = \gamma\vec{s}_1 + \delta\vec{s}_2$, and furthermore $\forall \gamma, \delta \exists \alpha, \beta$ so that $\alpha\vec{q}_1 + \beta\vec{q}_2 = \gamma\vec{s}_1 + \delta\vec{s}_2$. Now we have translated the statement in question into something purely mathematical that we can hope to prove. We've used the definition of spans — namely that the span of a list of vectors is the set of all linear combinations of those vectors. Something is in a span if it can be expressed as the appropriate linear combination.

The next question is to see how would we prove such statements based on what we know about the vectors in question. For the first direction, we know that if we could express both $\vec{q}_1$ and $\vec{q}_2$ as linear combinations of $\vec{s}_1$ and $\vec{s}_2$, we'd essentially be done. At that point, simply expanding out $\alpha\vec{q}_1 + \beta\vec{q}_2$ and combining terms would give us something in the desired form $\gamma\vec{s}_1 + \delta\vec{s}_2$. The identical argument could be used in the reverse direction if we could express both $\vec{s}_1$ and $\vec{s}_2$ as linear combinations of $\vec{q}_1$ and $\vec{q}_2$.

We first show that each vector in $\{\vec{q}_1, \vec{q}_2\}$ can be written as a linear combination of $\{\vec{s}_1, \vec{s}_2\}$. By construction, we first have that

$$\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|} \tag{5}$$

so $\vec{q}_1$ is a linear combination of $\{\vec{s}_1, \vec{s}_2\}$. And, unrolling the Gram-Schmidt algorithm,

$$\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} = \frac{\vec{s}_2 - (\vec{q}_1^\top \vec{s}_2)\vec{q}_1}{\left\|\vec{s}_2 - (\vec{q}_1^\top \vec{s}_2)\vec{q}_1\right\|} = \underbrace{\frac{1}{\left\|\vec{s}_2 - (\vec{q}_1^\top \vec{s}_2)\vec{q}_1\right\|}}_{\text{a scalar}} \vec{s}_2 - \underbrace{\frac{\vec{q}_1^\top \vec{s}_2}{\left\|\vec{s}_2 - (\vec{q}_1^\top \vec{s}_2)\vec{q}_1\right\| \|\vec{s}_1\|}}_{\text{another scalar}} \vec{s}_1 \tag{6}$$

so $\vec{q}_2$ is also a linear combination of $\{\vec{s}_1, \vec{s}_2\}$.

To complete the proof, we must now show that each vector in $\{\vec{s}_1, \vec{s}_2\}$ can be written as a linear combination of $\{\vec{q}_1, \vec{q}_2\}$ (the reverse direction). Indeed,

$$\vec{s}_1 = \|\vec{s}_1\| \vec{q}_1 \tag{7}$$

so $\vec{s}_1$ is a linear combination of $\{\vec{q}_1, \vec{q}_2\}$. And from the same unrolling as above, some algebra gets

$$\vec{s}_2 = \underbrace{\left\|\vec{s}_2 - (\vec{q}_1^\top \vec{s}_2)\vec{q}_1\right\|}_{\text{a scalar}} \vec{q}_2 + \underbrace{(\vec{q}_1^\top \vec{s}_2)}_{\text{another scalar}} \vec{q}_1. \tag{8}$$

Thus $\vec{s}_2$ is a linear combination of $\{\vec{q}_1, \vec{q}_2\}$ and this concludes the proof.

The important thing above is to understand how one proceeds systematically. When given a statement that you want to prove, you first need to state it mathematically. To state it, you generally need to first look at the types of the objects involved, and then leverage the definitions. Once you've stated things, to prove what you need to prove, you need to leverage what you know. In this case, because what we know is a procedure, we need to unroll the procedure effectively.

(d) **What would happen if $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ were *not* linearly independent, but rather $\vec{s}_1$ were a multiple of $\vec{s}_2$?**

Solution: If $\vec{s}_2$ is a multiple of $\vec{s}_1$, then $\vec{z}_2 = 0$. This means that the projection of $\vec{s}_2$ onto $\text{Span}(\{\vec{s}_1\})$ is just $\vec{s}_2$, so we have found an orthonormal basis for $\text{Span}(\{\vec{s}_1, \vec{s}_2\})$, in particular the basis $\{\vec{q}_1\}$. Hence, we can move onto $\vec{s}_3$ and continue the algorithm from there.

(e) Now given $\vec{q}_1$ and $\vec{q}_2$ in parts (a) and (b), **find $\vec{q}_3$ such that** $\text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$**, and $\vec{q}_3$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, and finally $\|\vec{q}_3\| = 1$.**

Solution: We want to project $\vec{s}_3$ onto $\text{Span}(\{\vec{q}_1, \vec{q}_2\})$. One matrix A such that $\text{Span}(\{\vec{q}_1, \vec{q}_2\}) = \text{Col}(A)$ is just $A = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$. Since $\{\vec{q}_1, \vec{q}_2\}$ are orthonormal, the projection matrix is

$$\begin{aligned}
A\left(A^\top A\right)^{-1} A^\top &= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \left(\begin{bmatrix} \vec{q}_1^\top \\ \vec{q}_2^\top \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \vec{q}_1^\top \\ \vec{q}_2^\top \end{bmatrix} \\
&= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \begin{bmatrix} \vec{q}_1^\top \vec{q}_1 & \vec{q}_1^\top \vec{q}_2 \\ \vec{q}_2^\top \vec{q}_1 & \vec{q}_2^\top \vec{q}_2 \end{bmatrix}^{-1} \begin{bmatrix} \vec{q}_1^\top \\ \vec{q}_2^\top \end{bmatrix} \\
&= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \vec{q}_1^\top \\ \vec{q}_2^\top \end{bmatrix} \\
&= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{q}_1^\top \\ \vec{q}_2^\top \end{bmatrix} \\
&= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \begin{bmatrix} \vec{q}_1^\top \\ \vec{q}_2^\top \end{bmatrix} \\
&= \vec{q}_1 \vec{q}_1^\top + \vec{q}_2 \vec{q}_2^\top.
\end{aligned}$$

Notice the key simplification here; the projection matrix converted into a vastly simplified form due to the orthonormality of our vectors. Proceeding,

$$A\left(A^\top A\right)^{-1} A^\top \vec{s}_3 = \vec{q}_1 \vec{q}_1^\top \vec{s}_3 + \vec{q}_2 \vec{q}_2^\top \vec{s}_3 = \left(\vec{q}_1^\top \vec{s}_3\right)\vec{q}_1 + \left(\vec{q}_2^\top \vec{s}_3\right)\vec{q}_2. \tag{9}$$

Note that the inner product of vectors yields a scalar, which can be safely moved around. To reiterate, note that this projection formula *only works if $\{\vec{q}_1, \vec{q}_2\}$ are orthonormal*. Otherwise, our projection matrix has a more complicated form. Then

$$\vec{z}_3 = \vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2 \tag{10}$$

is orthogonal to $\vec{q}_1$ and $\vec{q}_2$, hence orthogonal to any vector in $\text{Span}(\{\vec{q}_1, \vec{q}_2\})$. Normalizing, we have $\vec{q}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|}$.

(f) **[Practice] Confirm that** $\text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$**.**

Solution: We only have to show that $\vec{q}_3$ can be written as a linear combination of $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$, and that $\vec{s}_3$ can be written as a linear combination of $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$. Precisely, every vector that can be written as $\sum_{i=0}^{i=3} \alpha_i \vec{q}_i$ (this is its α-coordinate representation) has an equivalent representation in β-coordinates as

$\sum_{i=0}^{i=3} \beta_i \vec{s}_i$, and similarly any vector with β-coordinates has a corresponding equivalent representation in α-coordinates.

The rest follows from part (c), taking the coefficients of $\vec{s}_3$ or $\vec{q}_3$ to be 0 in every linear combination.

To write $\vec{q}_3$ as a linear combination of $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$, we unroll the Gram-Schmidt algorithm:

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{z}_3}{\|\vec{z}_3\|} = \frac{\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2}{\left\|\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2\right\|} \\ &= \underbrace{\frac{1}{\left\|\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2\right\|}}_{\text{a scalar}} \vec{s}_3 - \underbrace{\frac{\vec{s}_3^\top \vec{q}_1}{\left\|\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2\right\|}}_{\text{another scalar}} \vec{q}_1 \\ &\quad - \underbrace{\frac{\vec{s}_3^\top \vec{q}_2}{\left\|\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2\right\|}}_{\text{another scalar}} \vec{q}_2\end{aligned}$$

Since we have already shown $\vec{q}_1$ and $\vec{q}_2$ are linear combinations of vectors in $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$, we see that $\vec{q}_3$ is a linear combination of $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$.

To write $\vec{s}_3$ as a linear combination of $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$, we again unroll the algorithm, this time doing another simplification:

$$\begin{aligned}\vec{q}_3 &= \frac{\vec{z}_3}{\|\vec{z}_3\|} = \frac{\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2}{\left\|\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2\right\|} \\ \vec{s}_3 &= \underbrace{\left\|\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2\right\|}_{\text{a scalar}} \vec{q}_3 + \underbrace{\left(\vec{s}_3^\top \vec{q}_1\right)}_{\text{another scalar}} \vec{q}_1 + \underbrace{\left(\vec{s}_3^\top \vec{q}_2\right)}_{\text{another scalar}} \vec{q}_2.\end{aligned}$$

Hence $\vec{s}_3$ is a linear combination of $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ and the proof is complete.

2. Orthonormal Matrices and Projections

An orthonormal matrix, A, is a matrix whose columns, $\vec{a}_i$, are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = \vec{a}_j^\top \vec{a}_i = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to 1, $\|\vec{a}_i\| = 1$). This implies that $\|\vec{a}_i\|_2 = \langle \vec{a}_i, \vec{a}_i \rangle = \vec{a}_i^\top \vec{a}_i = 1$.

(a) When $A \in \mathbb{R}^{n\times m}$ and $n \geq m$ (i.e. for tall matrices), **show that if the matrix is orthonormal, then $A^\top A = I_{m\times m}$.**

Solution: We want to show $A^\top A = I_{M\times M}$. We proceed directly from the definition of matrix multiplication, using that the columnns of A are indexed by $\vec{a}_i$:

$$A^\top A = \begin{bmatrix} \vec{a}_1^\top \vec{a}_1 & \vec{a}_1^\top \vec{a}_2 & \dots & \vec{a}_1^\top \vec{a}_m \\ \vec{a}_2^\top \vec{a}_1 & \vec{a}_2^\top \vec{a}_2 & \dots & \vec{a}_2^\top \vec{a}_m \\ \vdots & \vdots & & \vdots \end{bmatrix} \tag{11}$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \end{bmatrix} \tag{12}$$

$$= I_{m\times m} \tag{13}$$

When $\vec{a}_i^\top \vec{a}_i = \|\vec{a}_i\|^2 = 1$ and when $i \neq j$, $\vec{a}_i^\top \vec{a}_j = 0$ because the column vectors are orthogonal.

(b) Again, suppose $A \in \mathbb{R}^{n\times m}$ where $n \geq m$ is an orthonormal matrix. **Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of A is now $AA^\top \vec{y}$.**

Solution: By applying the standard last squares projection result, we have that

$$A\vec{\hat{x}} = A\left(A^\top A\right)^{-1} A^\top \vec{y}, \tag{14}$$

Aside/Review: Note that there's an A matrix in front! This is a key distinction that is easy to miss; there is the solution to least squares, which is the vector $\vec{x}$. This tells us how much of each of the column vectors of A to take (weighted linear combination) to provide the best possible estimate to $\vec{y}$ (and this will not in general be equal to the actual $\vec{y}$). By multiplying by A, we get the actual esimation of the closest vector to $\vec{y}$ (as opposed to a vector containing the coefficients to weight each column vector by).

With this understanding, we can apply the result from part (a),

$$A\left(A^\top A\right)^{-1} A^\top \vec{y} = AIA^\top \vec{y} \tag{15}$$

$$= AA^\top \vec{y} \tag{16}$$

Now, we have shown that the projection of $\vec{y}$ onto the subspace spanned by the orthonormal columns of A simplifies to $AA^\top \vec{y}$.

(c) **Show if $A \in \mathbb{R}^{n\times n}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^N$.** *Hint: can you use the result of the previous subpart?*

Solution: As it turns out, the result we established in the previous subpart serves to very effectively and efficiently prove this statement, while remaining fully rigorous.

We first note that A is square and orthonormal. We know from part (a) that if $n = m$, the product $A^\top A$ yields $I_{n\times n}$. We therefore can say that A^T is A^{-1}. At this point, we are already done. A is invertible and so the columns of A form a basis for the space. Why? Because A^{-1} gives us the exact coordinates in this new basis for any point.

We could also continue by thinking about projections. In the previous part, we have a matrix $AA^\top$ that tells us how to project. But a square matrix times its inverse is the identity, no matter which order we multiply them. This means that the matrix $AA^\top$ must be the identity matrix (consider that for a square orthonormal matrix, $A^\top = A^{-1}$). Once we see that projection of any vector returns back the same vector, it must be that we are projecting onto a basis for the entire space.

Approach for the Skeptical: We want to show that the columns of A form a basis for $\mathbb{R}^n$. To do so, we need to show that:

- The columns are a set of n linearly independent vectors.
- Any vector $\vec{x} \in \mathbb{R}^n$ can be represented as a linear combination of the column vectors.

We already know we have n vectors by definition of A's dimensions, so first we will show they are linearly independent. We shall do this by showing that $A\vec{\beta} = \vec{0}$ implies that $\vec{\beta}$ can be only $\vec{0}$.

$$A\vec{\beta} = \vec{0} \tag{17}$$

$$\beta_1\vec{a}_1 + \ldots + \beta_N\vec{a}_n = \vec{0} \tag{18}$$

Then to exploit the properties of orthogonal vectors, we consider taking the inner product of each side of the above equation with $\vec{a}_i$.

$$\langle \vec{a}_i, \beta_1\vec{a}_1 + \ldots + \beta_N\vec{a}_N\rangle = \langle \vec{a}_i, \vec{0}\rangle = 0 \tag{19}$$

Now we apply the distributive property of the inner product (which is just a matrix multiplication) and the definition of orthonormal vectors,

$$\langle \vec{a}_i, \beta_1\vec{a}_1\rangle + \ldots + \langle \vec{a}_i, \beta_i\vec{a}_i\rangle + \ldots + \langle \vec{a}_i, \beta_N\vec{a}_N\rangle = 0 \tag{20}$$

$$0 + \ldots + \beta_i\left\langle \vec{a}_i, \vec{a}_j\right\rangle + \ldots + 0 = 0 \tag{21}$$

$$0 + \ldots + \beta_i\vec{a}_i^\top\vec{a}_i + \ldots + 0 = 0 \tag{22}$$

Because $\vec{a}_i^\top\vec{a}_i = 1$, $\beta_i = 0$ for the equation to hold. Then, since this is true for all i from 1 to n, all the elements of the vector beta must be zero ($\vec{\beta} = \vec{0}$). Because $\vec{x} = \vec{0}$ implies $\vec{\beta} = \vec{0}$, the columns of A are linearly independent.

Now, we will show that any vector $\vec{x} \in \mathbb{R}^n$ can be represented as a linear combination of the columns of A.

$$\vec{x} = A\vec{\beta} = \beta_1\vec{a}_1 + \ldots + \beta_N\vec{a}_N \tag{23}$$

Because we know that the n columns of A are linearly independent, then there exists A^{-1}. Applying the inverse to the equation above,

$$A^{-1}A\vec{\beta} = A^{-1}\vec{x} \tag{24}$$

$$\vec{\beta} = A^{-1}\vec{x}, \tag{25}$$

we find that there exists a unique β that allow us to represent any $\vec{x}$ as a linear combination of the columns of A.

Contributors:

- Regina Eckert.
- Druv Pai.
- Neelesh Ramachandran.

