EECS 16B Fall 2021 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 12A

The following notes are useful for this discussion: Note 19

1. Jacobians and Linear Approximation

Recall that for a scalar-valued function $f(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ with vector-valued arguments, we can linearize the function at $(\vec{x}_\star, \vec{y}_\star)$

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_\star,\vec{y}_\star) + (D_{\vec{x}} f)\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{x}-\vec{x}_\star) + (D_{\vec{y}} f)\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{y}-\vec{y}_\star). \tag{1}$$

where

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}, \tag{2}$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_k} \end{bmatrix}. \tag{3}$$

(a) When the function $\vec{f}(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a vector (rather than a scalar), we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them:

$$\vec{f}(\vec{x},\vec{y}) = \begin{bmatrix} f_1(\vec{x},\vec{y}) \\ f_2(\vec{x},\vec{y}) \\ \vdots \\ f_m(\vec{x},\vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_1 \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_1 \cdot (\vec{y}-\vec{y}_\star) \\ f_2(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_2 \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_2 \cdot (\vec{y}-\vec{y}_\star) \\ \vdots \\ f_m(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}} f_m \cdot (\vec{x}-\vec{x}_\star) + D_{\vec{y}} f_m \cdot (\vec{y}-\vec{y}_\star) \end{bmatrix} \tag{4}$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}} \vec{f} = \begin{bmatrix} D_{\vec{x}} f_1 \\ D_{\vec{x}} f_2 \\ \vdots \\ D_{\vec{x}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \tag{5}$$

and similarly

$$D_{\vec{y}} \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \tag{6}$$

Then, the linearization becomes

$$\vec{f}(\vec{x},\vec{y}) \approx \vec{f}(\vec{x}_\star,\vec{y}_\star) + (D_{\vec{x}} \vec{f})\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{x}-\vec{x}_\star) + (D_{\vec{y}} \vec{f})\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{y}-\vec{y}_\star). \tag{7}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **and** $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$**. Find** $D_{\vec{x}}\vec{f}$**, applying the definition above.**

Solution: Here, we have

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} 2x_1x_2 & x_1^2 \\ x_2^2 & 2x_1x_2 \end{bmatrix}. \tag{8}$$

(b) **Evaluate the approximation of** $\vec{f}$ **using** $\vec{x}_\star = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **at the point** $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$**, and compare with** $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$**.**

Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$**.**

Solution: Let $\delta = 0.01$. The true value is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} (2+\delta)^2(3+\delta) \\ (2+\delta)(3+\delta)^2 \end{bmatrix} = \begin{bmatrix} 12+16\delta+7\delta^2+\delta^3 \\ 18+21\delta+8\delta^2+\delta^3 \end{bmatrix}. \tag{9}$$

On the other hand, our approximation is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) \approx \vec{f}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + \begin{bmatrix} 12 & 4 \\ 9 & 12 \end{bmatrix} \cdot \begin{bmatrix} \delta \\ \delta \end{bmatrix} = \begin{bmatrix} 12+16\delta \\ 18+21\delta \end{bmatrix}. \tag{10}$$

Again, our approximation essentially removes the higher order terms of δ.
When we plug in $\delta = 0.01$, we have

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} 12.160701 \\ 18.210801 \end{bmatrix} \tag{11}$$

and our approximation is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} 12.16 \\ 18.21 \end{bmatrix}. \tag{12}$$

(c) Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x},\vec{y}) = \vec{x}\vec{y}^\top\vec{w}$. **Find** $D_{\vec{x}}\vec{f}$ **and** $D_{\vec{y}}\vec{f}$**.**

Solution: Here, recall that

$$\vec{f} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 & y_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1y_1w_1 + x_1y_2w_2 \\ x_2y_1w_1 + x_2y_2w_2 \end{bmatrix}. \tag{13}$$

Then,

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} y_1w_1 + y_2w_2 & 0 \\ 0 & y_1w_1 + y_2w_2 \end{bmatrix} \tag{14}$$

and

$$D_{\vec{y}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} x_1 w_1 & x_1 w_2 \\ x_2 w_1 & x_2 w_2 \end{bmatrix}. \tag{15}$$

We can also write

$$D_{\vec{x}}\vec{f} = \vec{y}^\top \vec{w} \cdot I \tag{16}$$

and

$$D_{\vec{y}}\vec{f} = \vec{x}\vec{w}^\top, \tag{17}$$

which can be derived by noticing that $\vec{y}^\top \vec{w} = \vec{w}^\top \vec{y}$.

(d) Continuing the above part, **find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.**

Solution: We have

$$\vec{f}(\vec{x}, \vec{y}) \approx \vec{f}(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}}\vec{f} \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}}\vec{f} \cdot (\vec{y} - \vec{y}_\star) \tag{18}$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 - 1 \\ y_2 - 1 \end{bmatrix} \tag{19}$$

(20)

Let's do an approximation of $\vec{f}\left(\begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix}, \begin{bmatrix} 1+\delta_3 \\ 1+\delta_4 \end{bmatrix}\right)$, then,

$$\vec{f}\left(\begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix}, \begin{bmatrix} 1+\delta_3 \\ 1+\delta_4 \end{bmatrix}\right) \approx \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 3 + 3\delta_1 + 2\delta_3 + \delta_4 \\ 3 + 3\delta_2 + 2\delta_3 + \delta_4 \end{bmatrix}.$$

We can compare with the true value

$$\begin{aligned} \vec{f}\left(\begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix}, \begin{bmatrix} 1+\delta_3 \\ 1+\delta_4 \end{bmatrix}\right) &= \begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix} \begin{bmatrix} 1+\delta_3 & 1+\delta_4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix} (3 + 2\delta_3 + \delta_4) \\ &= \begin{bmatrix} 3 + 3\delta_1 + 2\delta_3 + \delta_4 + 2\delta_1\delta_3 + \delta_1\delta_4 \\ 3 + 3\delta_2 + 2\delta_3 + \delta_4 + 2\delta_2\delta_3 + \delta_2\delta_4 \end{bmatrix}, \end{aligned} \tag{21}$$

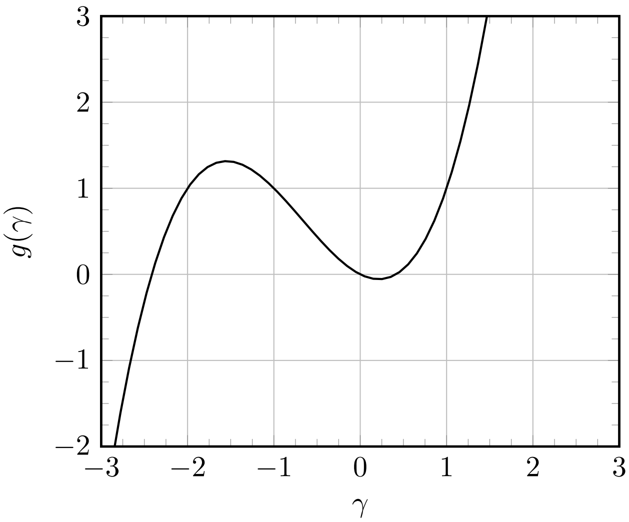
and we see that our approximation removes the second order δ terms $\delta_1\delta_3$, $\delta_1\delta_4$, $\delta_2\delta_3$ and $\delta_2\delta_4$.

2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix}-2\beta(t)+\gamma(t)\\ g(\gamma(t))+u(t)\end{bmatrix} = \vec{f}(\vec{x}(t), u(t)) \tag{22}$$

where $\vec{x}(t) = \begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix}$ and $g(\cdot)$ is a nonlinear function with the following graph:



The $g(\cdot)$ is the only nonlinearity in this system. We want to linearize this entire system around a operating point/equilibrium. Any point $x_\star$ is an operating point if $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{0}$.

(a) **If we have fixed $u_\star(t) = -1$, what values of γ and β will ensure $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{0}$?**

Solution: To find the equilibrium point, we'll start by finding the values for which $g(\gamma) + u^\star = g(\gamma) - 1 = 0$. In other words, we need to find values of γ such that $g(\gamma) = 1$. Although we don't have an equation for $g(\gamma)$, we can still find these points *graphically*, by using our graph. If we add a horizonal line at $g(\gamma) = 1$, we get the following:

Having done this, it looks like we'll have $f_2(\vec{x}, u^\star) = g(\gamma) - u^\star = 0$ for $\gamma = -2, \gamma = -1$, and $\gamma = 1$. Now we just need to find an β that sets $f_1(\vec{x}, u^\star) = -2\beta + \gamma = 0$ for each of these. Setting $\beta = \frac{1}{2} \cdot \gamma$ will do this.

With that, we have our three equilibrium points, namely

$$\vec{x}_1^\star = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \qquad \vec{x}_2^\star = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix} \qquad \vec{x}_3^\star = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}. \tag{23}$$

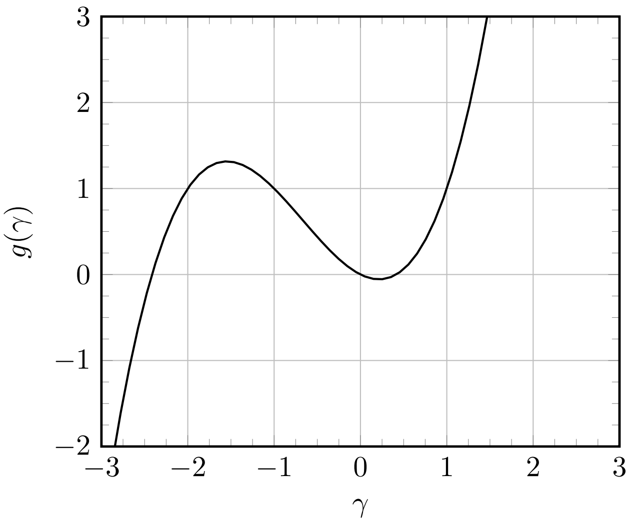
(b) Now that you have the three operating points, **linearize the system about the operating point** $(\vec{x}_3^\star, u_\star)$ **that has the largest value for** γ. Specifically, what we want is as follows. Let $\vec{\delta x}_i(t) = \vec{x}(t) - \vec{x}_i^\star$ for $i = 1, 2, 3$, and $\delta u(t) = u(t) - u_\star$. We can in principle write the *linearized system* for each operating point in the following form:

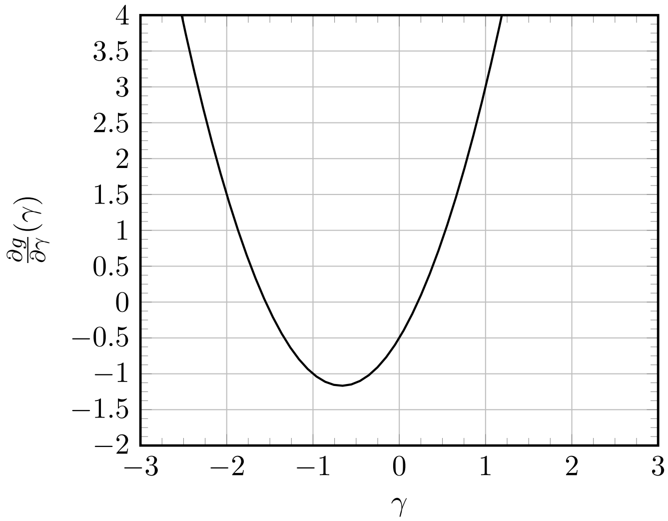
$$\text{(linearization about } (\vec{x}_i^\star, u_\star)) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\vec{\delta x}_i(t) = A_i \vec{\delta x}_i(t) + B_i \delta u(t) + \vec{w}_i(t) \tag{24}$$

where $\vec{w}_i(t)$ is a disturbance that also includes the approximation error due to linearization.

For this part, **find** A_i **and** B_i.

We have provided below the function $g(\gamma)$ and its derivative $\frac{\partial g}{\partial \gamma}$.





Solution: To linearize the system, we need to compute the two Jacobians

$$D_{\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial \beta} & \frac{\partial f_1}{\partial \gamma} \\ \frac{\partial f_2}{\partial \beta} & \frac{\partial f_2}{\partial \gamma} \end{bmatrix} \tag{25}$$

$$D_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u}, \end{bmatrix} \tag{26}$$

and evaluate them at the operating points that we found in the previous part. The Jacobian matrices evaluated at the operating points will be the A_i and B_i matrices.

If we work out the partial derivatives, we get

$$\frac{\partial f_1}{\partial \beta} = \frac{\partial}{\partial \beta}(-2\beta + \gamma) = -2 \tag{27}$$

$$\frac{\partial f_1}{\partial \gamma} = \frac{\partial}{\partial \gamma}(-2\beta + \gamma) = 1 \tag{28}$$

$$\frac{\partial f_2}{\partial \beta} = \frac{\partial}{\partial \beta}(g(\gamma) + u) = 0 \tag{29}$$

$$\frac{\partial f_2}{\partial \gamma} = \frac{\partial}{\partial \gamma}(g(\gamma) + u) = \frac{\partial g}{\partial \gamma} \tag{30}$$

$$\frac{\partial f_1}{\partial u} = \frac{\partial}{\partial u}(-2\beta + \gamma) = 0 \tag{31}$$

$$\frac{\partial f_2}{\partial u} = \frac{\partial}{\partial u}(g(\gamma) + u) = 1, \tag{32}$$

$$\tag{33}$$

which gives

$$D_{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & \frac{\partial g}{\partial \gamma} \end{bmatrix} \tag{34}$$

$$D_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{35}$$

It turns out that the only part of $D_{\vec{x}}$ and D_u that depends on the operating point is $\partial g / \partial \gamma$, and we can read these off of the given graph. The relevant values are

$$\left.\frac{\partial g}{\partial \gamma}\right|_{\gamma=-2} = 1.5 \tag{36}$$

$$\left.\frac{\partial g}{\partial \gamma}\right|_{\gamma=-1} = -1 \tag{37}$$

$$\left.\frac{\partial g}{\partial \gamma}\right|_{\gamma=2} = 3, \tag{38}$$

which correspond to $\vec{x}_1^\star$,$\vec{x}_2^\star$, and $\vec{x}_3^\star$, respectively. Finally, this gives

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1.5 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{39}$$

$$A_2 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{40}$$

$$A_3 = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{41}$$

$$\tag{42}$$

(c) **Which of the operating points are *stable*? Which are *unstable?***

Solution: To assess the stability or instability of each operating point, we need to find the eigenvalues of each linearization. Since A_1, A_2, and A_3 are all *upper triangular*, their eigenvalues are just the two entries along their diagonals. So, the linearization will be stable if both diagonal entries are negative (remember, these are *continuous-time* systems), and unstable if they aren't both negative. This means that:

- $\vec{x}_1^\star$ is *unstable*, since the eigenvalues of A_1 are -2 and 1.5;
- $\vec{x}_2^\star$ is *stable*, since the eigenvalues of A_2 are -2 and -1;
- $\vec{x}_3^\star$ is *unstable*, since the eigenvalues of A_3 are -2 and 3.

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