EECS 16B Fall 2021 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 15A

In this discussion we review Spring 2021 Final. For each group the bold questions are the ones that we'll over first, and the rest if time permits.

(a) **Slot 1 : 11am - 12pm**

- 204 Wheeler (Krishna) : **Q3, Q4**, Q5
- 222 Wheeler (Gavin) : **Q2, Q6**, Q1
- 241 Cory (Jichan) : **Q2, Q5**, Q3

(b) **Slot 2 : 12pm - 1pm**

- 103 Moffitt (Michael) : **Q3, Q4**, Q5 (extended section : 12pm-2pm)
- 108 Wheeler (Gavin) : **Q2, Q6**, Q1
- 219 Dwinelle (Jichan) : **Q2, Q5**, Q3
- 3109 Etcheverry (Manav) : **Q2, Q5**, Q3

(c) **Slot 3 : 2pm - 3pm**

- 108 Wheeler (Krishna) : **Q3, Q4**, Q5
- 20 Wheeler (Gavin) : **Q2, Q6**, Q1 (extended section : 2pm-4pm)
- 3111 Etcheverry (Manav) : **Q2, Q5**, Q3

(d) **Slot 4 : 5pm - 6pm**

- 170 SOCS (Michael) : **Q3, Q4**, Q5
- 106 Moffitt (Jichan) : **Q2, Q5**, Q3

1. Potpourri

(a) Consider:

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{1}$$

Run Gram-Schmidt on these vectors in this order (that is, start with $\vec{v}_1$ then $\vec{v}_2$), and extend this set to form an orthonormal basis for $\mathbb{R}^3$. Show your work.

Solution: The concepts needed to solve this problem were explored in Note 10, Homework 10 Q2, and Discussion 9B.

We run the Gram-Schmidt process:

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\vec{v}_1}{5} = \begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}. \tag{2}$$

$$\vec{z}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{q}_1 \rangle \vec{q}_1 = \vec{v}_2 - \frac{3}{5}\vec{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 9/25 \\ 12/25 \end{bmatrix} = \begin{bmatrix} 0 \\ 16/25 \\ -12/25 \end{bmatrix} \tag{3}$$

$$\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} = \frac{\vec{v}_2}{4/5} = \begin{bmatrix} 0 \\ 4/5 \\ -3/5 \end{bmatrix}. \tag{4}$$

To find a suitable final vector, we just note that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ completes the orthonormal basis. We can also get this through Gram-Schmidt by adjoining the standard basis $\vec{v}_3 = \vec{e}_1$, $\vec{v}_4 = \vec{e}_2$, $\vec{v}_5 = \vec{e}_3$ (where $\vec{e}_i$ corresponds to each of our standard basis vectors) to our list:

$$\vec{z}_3 = \vec{v}_3 - \langle \vec{v}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{v}_3, \vec{q}_2 \rangle \vec{q}_2 = \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

$$\implies \vec{q}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$\vec{z}_4 = \vec{v}_4 - \langle \vec{v}_4, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{v}_4, \vec{q}_2 \rangle \vec{q}_2 - \langle \vec{v}_4, \vec{q}_3 \rangle \vec{q}_3 = \vec{0} \tag{7}$$

$$\vec{z}_5 = \vec{v}_5 - \langle \vec{v}_5, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{v}_5, \vec{q}_2 \rangle \vec{q}_2 - \langle \vec{v}_5, \vec{q}_3 \rangle \vec{q}_3 = \vec{0}. \tag{8}$$

(b) Consider the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which by the Spectral Theorem has an eigendecomposition $A = WDW^{-1}$ where

$$W = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Write the SVD of $A = U\Sigma V^\top$ and identify U, Σ, V.

Solution: The concepts needed to solve this problem were explored in Note 13, Homework 11 Q3, and Discussion 11A/B and 12A/B.

The natural way to solve this problem is by computing the SVD as you have done many times in HW and discussion.

There is also a quick way to do it by observing that symmetric matrices inspired our SVD decomposition – and for symmetric matrices, the EVD and SVD are the same up to sign changes in the eigenvalues and eigenvectors and reordering of the eigenvalues to be in decreasing order. Since the eigenvalues are all positive, then we don't need any sign changes. Since the eigenvalue matrix D already has decreasing eigenvalues, we don't need to reorder it either. Thus the SVD of $A = U\Sigma V^\top$ is $U = W, \Sigma = D, V = W$.

(c) In digital design, we often use 'synchronous' circuits, i.e. circuits which evaluate when a clock signal transitions from 0 to V_{DD}. One such implementation, called domino CMOS logic, is shown in Figure 1. Initially $V_{\text{clk}} = 0$ ('reset phase') for a long time, so the output node is high, i.e. $V_{\text{out}} = V_{DD}$ and the capacitor is fully charged, regardless of the values of V_A and V_B. We want to complete the Truth Table 1 during the 'evaluation phase'. **For cases (ii) and (iv), when V_{clk} transitions from 0 to V_{DD} and V_A and V_B are equal to the values specified in the table, what is V_{out}? Justify your answer.**

Note that if all transistors connected to the output node are switched off, then the capacitor C at the output node 'holds' the voltage since there is no charging / discharging path in that case.

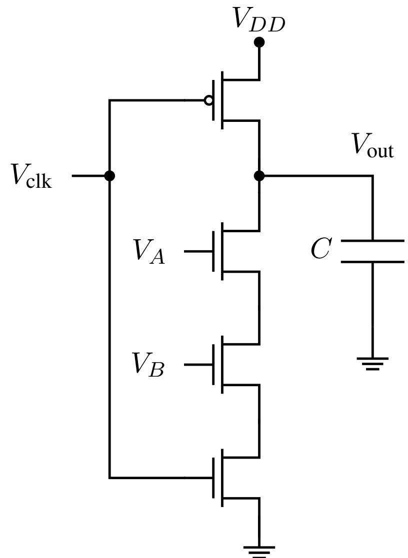


Figure 1: Domino Logic Gate

Case	V_{clk}	V_A	V_B	V_{out}
(i)	$0 \to V_{DD}$	0	0	$V_{DD} \to V_{DD}$
(ii)	$0 \to V_{DD}$	0	V_{DD}	$V_{DD} \to$ **?**
(iii)	$0 \to V_{DD}$	V_{DD}	0	$V_{DD} \to V_{DD}$
(iv)	$0 \to V_{DD}$	V_{DD}	V_{DD}	$V_{DD} \to$ **?**

Table 1: Truth Table

Solution: The concepts needed to solve this problem were explored in Note 1B, Discussion 9A Q2, and Discussion 2A Q3.

Since $V_{\text{clk}} = V_{DD}$, the PMOS is switched off and the NMOS closest to ground is switched on.
In Case (ii), $V_A = 0$, which means the corresponding NMOS is switched off, so the entire NMOS network is 'off'. Hence the output node is floating, so it stays at the voltage stored in the capacitor, hence $V_{\text{out}} = V_{DD}$.
In Case (iv), is $V_A = V_{DD}$ and $V_B = V_{DD}$, so all the NMOS are switched on and the output node can discharge to ground. Hence $V_{\text{out}} = 0$.

2. Analog Signal Processing

In this problem, we will study an example of one of the most common applications in signal processing: removing noise and amplifying the desired signal in a receiver.

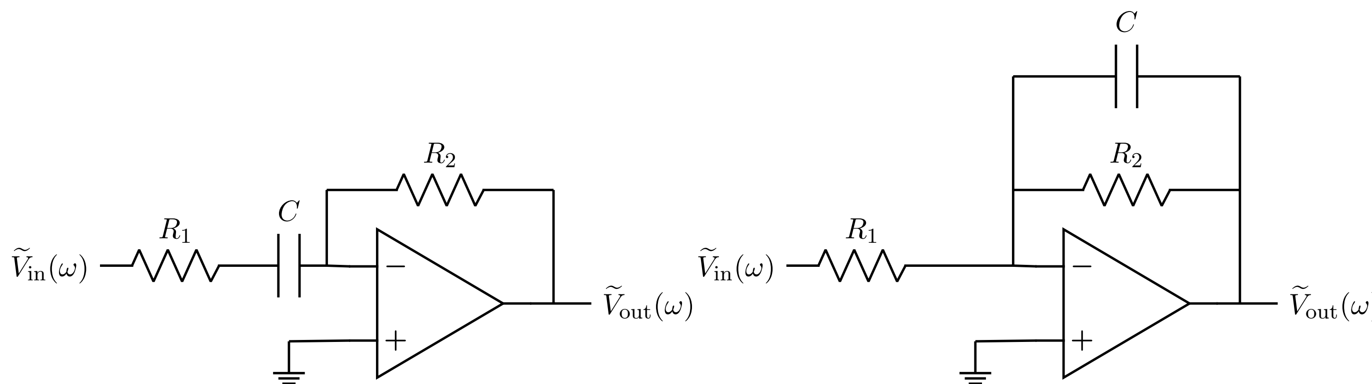
In 16B we have learned about filters, so we can selectively remove specific noise frequency bands. Assume that we have a low frequency desired signal $s(t) = \cos(\omega_{\text{sig}} t)$, where $\omega_{\text{sig}} = 10\frac{\text{rad}}{\text{s}}$, and a high frequency noise $n(t) = 2\cos(\omega_{\text{noise}} t)$, where $\omega_{\text{noise}} = 1000\frac{\text{rad}}{\text{s}}$, at the receiver input. We wish to amplify the desired signal and also reject the noise.

(a) Let's first attempt to use a low-pass filter to achieve this goal. Since we wish to amplify the desired signal, we need to use a low-pass filter with gain > 1 (i.e. use an amplifier combined with a filter). Assume that the op-amps are ideal and follow the golden rules.

i. **Derive a transfer function for the filter configuration in Figure 2a.** Show your work.

ii. **Derive a transfer function for the filter configuration in Figure 2b.** Show your work.

iii. Out of the two filter configurations in Figure 2, **which one is the low-pass filter? Justify your answer.**



(a) Config 1 **(b)** Config 2

Figure 2: Active filter receiver configurations

Solution: The concepts needed to solve this problem were explored in Note 5, Homework 7 Q4, and EECS 16A.

i. For the first configuration in Figure 2a,

$$
\begin{aligned}
Z_{in,1} &= R_1 + \frac{1}{j\omega C} \\
Z_{fb,1} &= R_2 \\
H_1(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)} &= -\frac{Z_{fb,1}}{Z_{in,1}} \\
&= -\frac{R_2}{R_1 + \frac{1}{j\omega C}} \\
&= -\frac{j\omega C R_2}{1 + j\omega C R_1} = -\frac{R_2}{R_1} \cdot \frac{1}{1 - \frac{j}{\omega C R_1}}
\end{aligned}
$$

Notice, that this transfer function has a gain component $\left(-\frac{R_2}{R_1}\right)$ and a high-pass filter component $\left(\frac{1}{1-\frac{j}{\omega C R_1}}\right)$.

ii. For the second configuration in Figure 2b,

$$\begin{aligned} Z_{in,2} &= R_1 \\ Z_{fb,2} &= R_2 || \frac{1}{j\omega C} = \frac{R_2}{1 + j\omega C R_2} \\ H_2(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)} &= -\frac{Z_{fb,2}}{Z_{in,2}} \\ &= -\frac{R_2}{R_1} \cdot \frac{1}{1 + j\omega C R_2} \end{aligned}$$

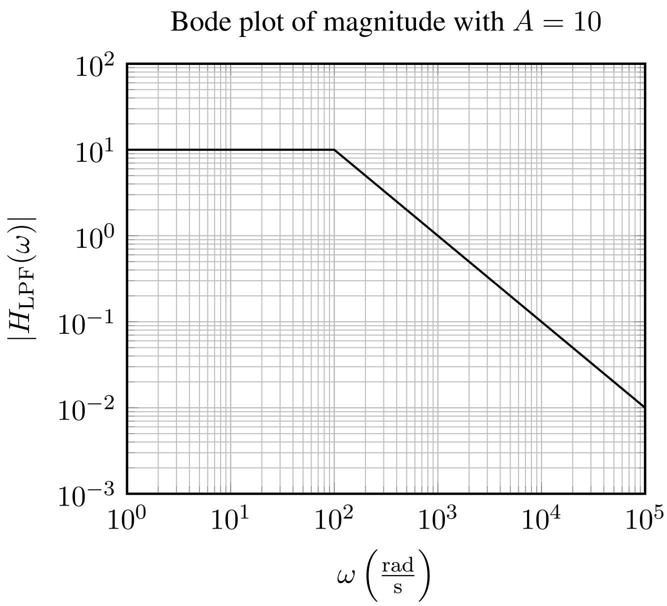
Notice, that this transfer function has a gain component $\left(-\frac{R_2}{R_1}\right)$ and a low-pass filter component $\left(\frac{1}{1+j\omega C R_2}\right)$.

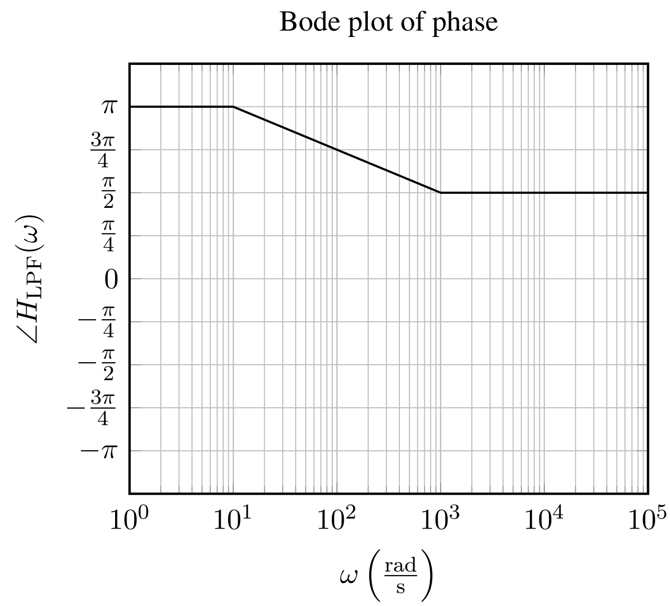
iii. Analyzing the transfer functions of the two configurations we see that Config 2 represents a low-pass filter. Example justification answers - Config 2 transfer function has a single pole. Looking at frequency extremes $\omega \to 0$ and $\omega \to \infty$ we conclude that the Config 2 transfer function represents a low-pass filter.

(b) Suppose that the transfer function of the low-pass filter with gain from part (a) was $H_{\text{LPF}}(\omega) = -\frac{A}{1+j\frac{\omega}{\omega_c}}$, where the cutoff frequency frequency is $\omega_c = 100\frac{\text{rad}}{\text{s}}$ and the gain is $A = 10$. The Bode plots for the low-pass filter with gain are shown below. Read-off the numerical values corresponding to the appropriate points on the Bode plots.

i. What are the magnitude and phase of the filter output signal when the input into the filter is $s(t) = \cos\big(\omega_{\text{sig}} t\big)$, where $\omega_{\text{sig}} = 10\frac{\text{rad}}{\text{s}}$? **Derive the time domain expression for the filter output signal.**

ii. What are the magnitude and phase of the filter output signal when the input into the filter is $n(t) = 2\cos(\omega_{\text{noise}} t)$, where $\omega_{\text{noise}} = 1000\frac{\text{rad}}{\text{s}}$? **Derive the time domain expression for the filter output signal.**





Solution: The concepts needed to solve this problem were explored in Note 5, Note 6, and Homework 7 Q3.

i. From the Bode plots, we can read that $H_{\mathrm{LPF}}(\omega_{\mathrm{sig}}) = 10e^{j\pi}$. Hence

$$\begin{aligned} s(t) &= \cos(10t) \\ \implies \widetilde{S} &= 0.5e^{j0} \\ \implies \widetilde{S_1} &= \widetilde{S} \cdot H_{\mathrm{LPF}}(\omega_{\mathrm{sig}}) \\ &= 5e^{j\pi} \\ \implies s_1(t) &= 10\cos(10t + \pi) = -10\cos(10t) \end{aligned}$$

Alternatively, realize that the transfer function directly affects the signal amplitude and phase to write the time domain answer $s_1(t) = 10\cos(10t + \pi) = -10\cos(10t)$.

ii. From the Bode plots, we can read that $H_{\mathrm{LPF}}(\omega_{\mathrm{noise}}) = e^{j\frac{\pi}{2}}$. Hence

$$\begin{aligned} n(t) &= 2\cos(1000t) \\ \implies \widetilde{N} &= e^{j0} \\ \implies \widetilde{N_1} &= \widetilde{N} \cdot H_{\mathrm{LPF}}(\omega_{\mathrm{noise}}) \\ &= e^{j\frac{\pi}{2}} \\ \implies n_1(t) &= 2\cos\left(1000t + \frac{\pi}{2}\right) = -2\sin(1000t) \end{aligned}$$

Alternatively, realize that the transfer function directly affects the noise amplitude and phase to write the time domain answer $n_1(t) = 2\cos\left(1000t + \frac{\pi}{2}\right) = -2\sin(1000t)$.

(c) We wish to have the signal be more amplified with respect to the noise. One approach is to cascade two copies of the filter $H_{\mathrm{LPF}}(\omega)$ to make a second-order low-pass filter with gain. Note that it is not necessary to put a unity gain buffer between the two filters, because the V_{out} loading **does not affect** the behavior of this specific filter configuration.

i. **Derive the transfer function** $H_{\mathrm{casc}}(\omega)$ **of the second-order low-pass filter** by cascading 2 of the first order transfer function $H_{\text{LPF}}(\omega) = -\frac{A}{1+j\frac{\omega}{\omega_c}}$ from part (b) with $\omega_c = 100\frac{\text{rad}}{\text{s}}$ and $A = 10$. **Show your work.**

ii. **Sketch the Bode magnitude and phase plots of $H_{\text{casc}}(\omega)$ on the charts in your answer template.**
(Hint: Pay attention to the direction of the slopes.)

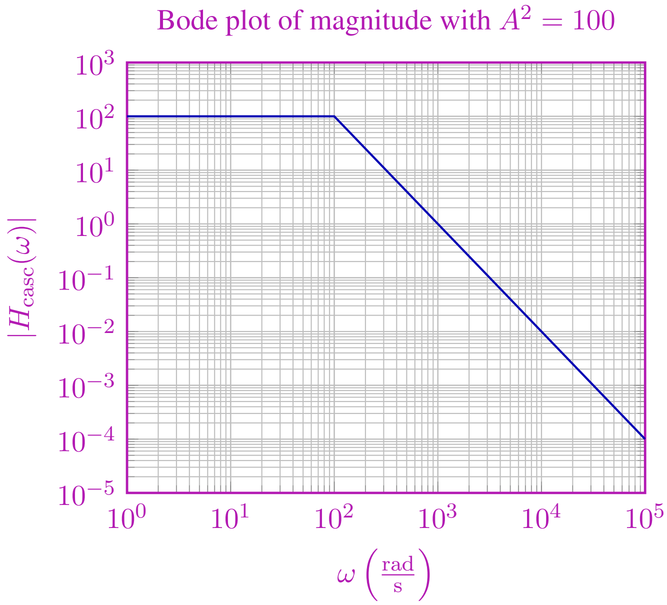
Solution: The concepts needed to solve this problem were explored in Note 5, Note 6, and Homework 7 Q3.

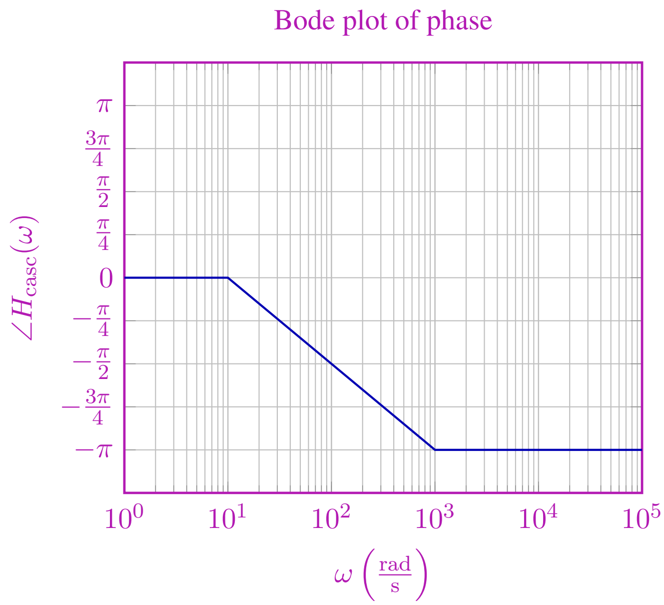
i. Since there is no loading effect, the transfer function of the cascaded filter is

$$\begin{aligned} H_{\text{casc}}(\omega) &= H_{\text{LPF}}^2(\omega) \\ &= \frac{A^2}{(1+j\frac{\omega}{\omega_c})^2} \end{aligned}$$

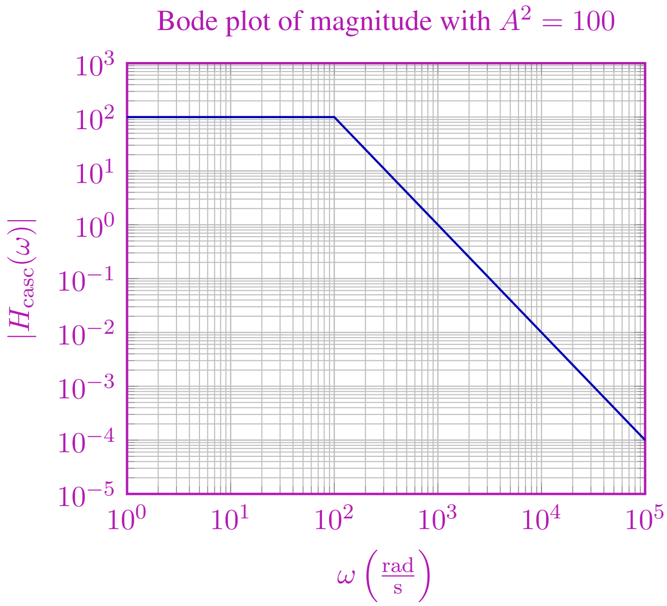
ii. The Bode magnitude plot satisfies the following properties:
- At frequencies $\omega \leq \omega_c = 100\frac{\text{rad}}{\text{s}}$, magnitude is $A^2 = 100$.
- At frequencies $\omega > \omega_c = 100\frac{\text{rad}}{\text{s}}$, magnitude drops by $100\times$ per power of 10 of ω.

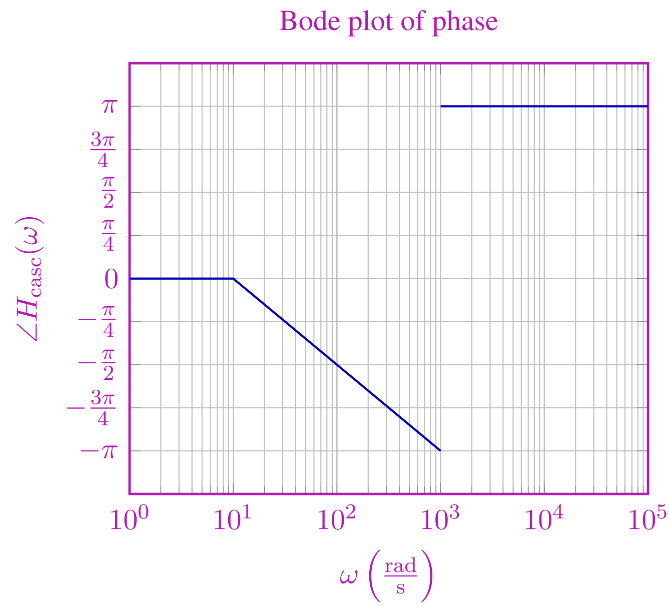
The Bode phase plot satisfies the following properties:
- At frequencies $\omega \leq \frac{\omega_c}{10} = 10\frac{\text{rad}}{\text{s}}$, phase is $0\,\text{rad} = 0°$.
- At frequencies $\omega \geq 10\omega_c = 1000\frac{\text{rad}}{\text{s}}$, phase is $-\pi$ rad $= -180°$.
- At $\omega = \omega_c = 100\frac{\text{rad}}{\text{s}}$, phase is $-\frac{\pi}{2}$ rad $= -90°$.





Alternatively the phase plot can also be drawn as follows, because $-\pi$ rad $= \pi$ rad.


Bode plot of magnitude with $A^2 = 100$
$|H_{\text{casc}}(\omega)|$
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^0 10^1 10^2 10^3 10^4 10^5
$\omega\left(\frac{\text{rad}}{\text{s}}\right)$


Bode plot of phase
$\angle H_{\text{casc}}(\omega)$
π
$\frac{3\pi}{4}$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
0
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$
$-\frac{3\pi}{4}$
$-\pi$
10^0 10^1 10^2 10^3 10^4 10^5
$\omega\left(\frac{\text{rad}}{\text{s}}\right)$

(d) Our implementation of the cascaded second-order filter from part (c) uses 2 op-amps. Can we get even more noise attenuation by using a single op-amp? One approach is to use a Notch filter that ideally completely rejects the noise.

Let's consider the cascade of an LC Notch filter with a non-inverting amplifier in Figure 3. We wish to have a notch at the noise frequency so that the noise $n(t) = 2\cos(\omega_{\text{noise}}t)$, where $\omega_{\text{noise}} = 1000\frac{\text{rad}}{\text{s}}$, is completely rejected, while the the signal $s(t) = \cos\big(\omega_{\text{sig}}t\big)$, where $\omega_{\text{sig}} = 10\frac{\text{rad}}{\text{s}}$, is amplified.

i. **Derive the transfer function $H_{\text{notch}}(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ of the filter in Figure 3.** Assume that the op-amp is ideal and follows the golden rules. Show your work.

ii. Using $C = 0.5\,\text{mF}$, **find the inductance value L so that the notch (i.e. the frequency at which the magnitude of the transfer function is 0) is at the noise frequency $\omega_{\text{noise}} = 1000\frac{\text{rad}}{\text{s}}$.** Show your work.

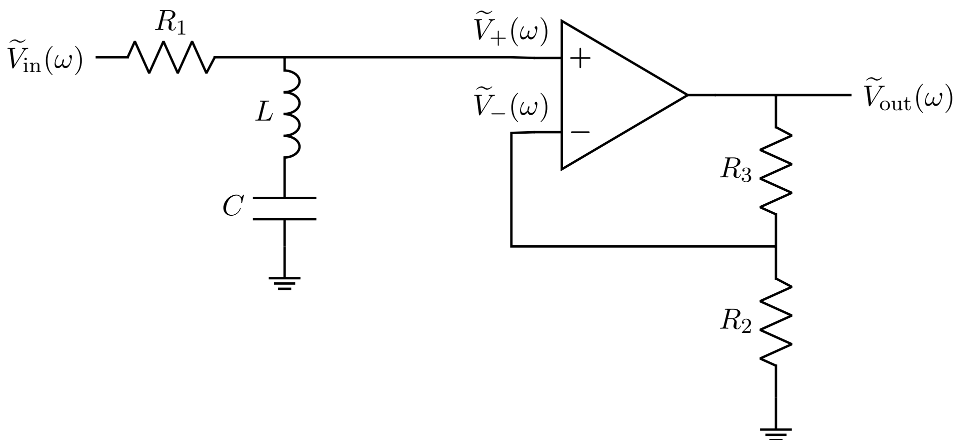


Figure 3: LC Notch filter and non-inverting amplifier

Solution: The concepts needed to solve this problem were explored in Lecture 6A and Homework 6 Q7.

i. The overall circuit in Fig 3 is the cascade of a voltage divider from $\widetilde{V}_{\text{in}}(\omega)$ to $\widetilde{V}_+(\omega)$ and a non-inverting amplifier from $\widetilde{V}_+(\omega)$ to $\widetilde{V}_{\text{out}}(\omega)$. The transfer function is derived as follows (op-amp in negative feedback implying $\widetilde{V}_+(\omega) = \widetilde{V}_-(\omega)$) :

$$\begin{aligned}
\frac{\widetilde{V}_+(\omega)}{\widetilde{V}_{\text{in}}(\omega)} &= \frac{j\omega L + \frac{1}{j\omega C}}{R_1 + j\omega L + \frac{1}{j\omega C}} \\
&= \frac{j\omega(L - \frac{1}{\omega^2 C})}{R_1 + j\omega(L - \frac{1}{\omega^2 C})} \\
\frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_+(\omega)} = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_-(\omega)} &= \frac{R_2 + R_3}{R_2} \\
\implies H_{\text{notch}}(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)} &= \left(1 + \frac{R_3}{R_2}\right) \cdot \frac{j\omega(L - \frac{1}{\omega^2 C})}{R_1 + j\omega(L - \frac{1}{\omega^2 C})} \\
&= \left(1 + \frac{R_3}{R_2}\right) \cdot \frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega C R_1}
\end{aligned}$$

ii. We wish to place the notch at the noise frequency. The notch is at the frequency where the magnitude of the transfer function is 0, i.e. magnitude of the numerator is 0. Hence

$$\begin{aligned}
|H_{\text{notch}}(\omega_{\text{noise}})| &= 0 \\
\implies L - \frac{1}{\omega_{\text{noise}}^2 C} &= 0 \\
\implies L &= \frac{1}{\omega_{\text{noise}}^2 C} \\
&= \frac{1}{1000^2 \times 0.5 \times 10^{-3}} = 2 \times 10^{-3}
\end{aligned}$$

Therefore the inductance is $L = 2\,\text{mH}$.

3. Optimization and Singular Values

We are going to focus on a special optimization problem that is related to the underlying structure of the SVD. More specifically, we want to solve for s in the following maximization problem

$$s = \max_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|^2}{\|\vec{x}\|^2}. \tag{9}$$

Here, we have $A \in \mathbb{R}^{m \times n}$. Let $m > n$ so that A is a tall matrix and $\operatorname{rank}(A) = n$. Let the full SVD be given by $A = U\Sigma V^\top$.

Define $\vec{x}^* \in \mathbb{R}^n$ to be the optimal vector that achieves the maximum in equation (9). That is,

$$\vec{x}^* = \operatorname*{argmax}_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|^2}{\|\vec{x}\|^2}, \tag{10}$$

$$s = \frac{\|A\vec{x}^*\|^2}{\|\vec{x}^*\|^2}. \tag{11}$$

(a) We start by attempting to simplify the optimization problem. **Prove that for any $\vec{x}$, we have $\|A\vec{x}\| = \|\Sigma V^\top \vec{x}\|$.** Note that you must justify and explain every step for full credit, just equations without an explanation may not be awarded full credit.

Solution: The concepts needed to solve this problem were explored in Note 13, Discussion 11A, and Homework 12 Q4b.

We can directly plug in the SVD of A into the left hand side:

$$\begin{aligned} \|A\vec{x}\| &= \|U\Sigma V^\top \vec{x}\| \\ &= \|U(\Sigma V^\top \vec{x})\| \\ &= \|\Sigma V^\top \vec{x}\|. \end{aligned}$$

We used the fact that U is an orthonormal matrix and therefore preserves the norm of any vector. This was proven in lecture, and also in HW 11 Q2(c).

Alternative Solution:

Alternatively, we can use the inner product definition of the norm in order to simplify the expression. Recall that $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\vec{x}^\top \vec{x}}$. Using this fact, we can write out

$$\begin{aligned} \|A\vec{x}\| &= \sqrt{(A\vec{x})^\top (A\vec{x})} \\ &= \sqrt{(U\Sigma V^\top \vec{x})^\top (U\Sigma V^\top \vec{x})} \\ &= \sqrt{\vec{x}^\top V \Sigma^\top (U^\top U) \Sigma V^\top \vec{x}}. \end{aligned}$$

We use the fact that U is an orthonormal matrix, implying that $U^\top U = I$. Substituting this back in gives us

$$
\begin{aligned}
\|A\vec{x}\| &= \sqrt{\vec{x}^\top V \Sigma^\top \Sigma V^\top \vec{x}} \\
&= \sqrt{(\Sigma V^\top \vec{x})^\top (\Sigma V^\top \vec{x})} \\
&= \|\Sigma V^\top \vec{x}\|.
\end{aligned}
$$

(b) Using a change of variables, we can in fact turn our original maximization problem into

$$s = \max_{\|\vec{w}\| \neq 0} \frac{\|\Sigma \vec{w}\|^2}{\|\vec{w}\|^2}. \tag{12}$$

Find the correct change of variables that relates $\vec{x}$ and $\vec{w}$ and show that optimization problems (9) and (12) are equivalent.

Hint: The change of variables you are looking for can also be thought of as a change of basis.

Solution: The concepts needed to solve this problem were explored in Lecture 11B and Homework 12 Q4.

Plugging in the result from part (a) into the original maximization problem yields

$$s = \max_{\|\vec{x}\| \neq \vec{0}} \frac{\|\Sigma V^\top \vec{x}\|^2}{\|\vec{x}\|^2}.$$

Seeing what the maximization problem transforms into, we see we want to choose $V^\top \vec{x} = \vec{w}$. This is since multiplication by $V^\top$ won't change the norm of a vector due to it being orthonormal. So, the problem becomes

$$
\begin{aligned}
s &= \max_{\|V\vec{w}\| \neq \vec{0}} \frac{\|\Sigma \vec{w}\|^2}{\|V\vec{w}\|^2} \\
&= \max_{\|\vec{w}\| \neq \vec{0}} \frac{\|\Sigma \vec{w}\|^2}{\|\vec{w}\|^2}.
\end{aligned}
$$

(c) **Let σ_1 be the largest singular value of matrix A. Find a $\vec{w}^*$, such that $\|\Sigma \vec{w}^*\|^2 = \sigma_1^2 \|\vec{w}^*\|^2$. Justify your answer.**

Solution: The concepts needed to solve this problem were explored in Note 4, Homework 12 Q3 and Q4.

We first define $\vec{e_i}$ (the i^{th} standard basis vector) as the vector with a 1 in the i^{th} entry and zero everywhere else (i.e. the i^{th} column of the identity matrix). If $\vec{w}^* = c\vec{e}_1 = \begin{bmatrix} c & 0 & \dots & 0 \end{bmatrix}^\top$ for some constant c, then

$$||\Sigma \vec{w}^*||^2 = ||\Sigma c \vec{e}_1||^2 = \sigma_1^2 c^2 = \sigma_1^2 ||\vec{w}^*||^2$$

Part (d) will tell us that for all $\vec{w}$, $\frac{||\Sigma \vec{w}||}{||\vec{w}||}^2 \leq \sigma_1^2$. This means that we have found the $\vec{w^*}$ that achieves the upper bound. So, the value of s from (9) must be equal to σ_1^2.

(d) **Prove that for all $\vec{w}$ we have $\|\Sigma \vec{w}\|^2 \leq \sigma_1^2 \|\vec{w}\|^2$. Show your work.**

Hint: Remember that Σ has n non-zero entries $\sigma_1 \geq \sigma_2 ... \geq \sigma_n$ along the diagonal, and all other entries are zero.

Solution: The concepts needed to solve this problem were explored in Note 4, Homework 12 Q3 and Q4.

We first note that σ_1 is the largest singular value, so it is greater than or equal to all σ_i. We start by rewriting $\|\Sigma\vec{w}\|^2$ as a summation,

$$||\Sigma\vec{w}||^2 = \sum_{i=1}^{n} \sigma_i^2 w_i^2.$$

Next, we use the fact that σ_1 is greater than or equal to all of the σ_i and invoke the inequality,

$$\sum_{i=1}^{n} \sigma_i^2 w_i^2 \leq \sum_{i=1}^{n} \sigma_1^2 w_i^2.$$

Finally, we can pull out the common σ_1^2 from the summation and substitute $\sum_{i=1}^{n} w_i^2$ as the norm-squared of $\vec{w}$,

$$\sum_{i=1}^{n} \sigma_1^2 w_i^2 = \sigma_1^2 \sum_{i=1}^{n} w_i^2 = \sigma_1^2 ||\vec{w}||^2,$$

as desired.

4. I bet Cal will win this year

As huge fans of the Big Game, you and your friend want to bet on whether Cal or Stanford will win this year. You want to predict this year's result by analyzing historical records. Therefore, you decide to model this as a binary classification problem and do PCA for dimension reduction on the data you collected. The "$+1$" class represents victories of Cal and "-1" represents victories of Stanford.

After some research, you obtained a data matrix $A \in \mathbb{R}^{n \times d}$,

$$A = \begin{bmatrix} - & \vec{x}_1^\top & - \\ - & \vec{x}_2^\top & - \\ & \vdots & \\ - & \vec{x}_n^\top & - \end{bmatrix} \tag{13}$$

where each of the n rows $\vec{x}_i^\top$ denotes a game and each of the d columns of A contains information of a possibly relevant factor of the games (weather, location, date, air quality, etc).

(a) Let the full SVD of $A = U\Sigma V^\top$, where A is given in eq. (13).

You project your data along $\vec{v}_1$ and $\vec{v}_2$ (the first two principal components along the rows), and for comparison you also project your data along two randomly chosen directions $\vec{w}_1$ and $\vec{w}_2$ as well. You get the two pictures in Figure 4, but you forgot to label the axes. Of the two figures below, which one is the projection onto the principal components and which one is the projection onto the random directions? **Match axes (i), (ii), (iii), (iv) to $\vec{w}_1, \vec{w}_2, \vec{v}_1$, and $\vec{v}_2$, and justify your answer.**

Note that there may be multiple correct matchings; you only need to find and justify one of them.

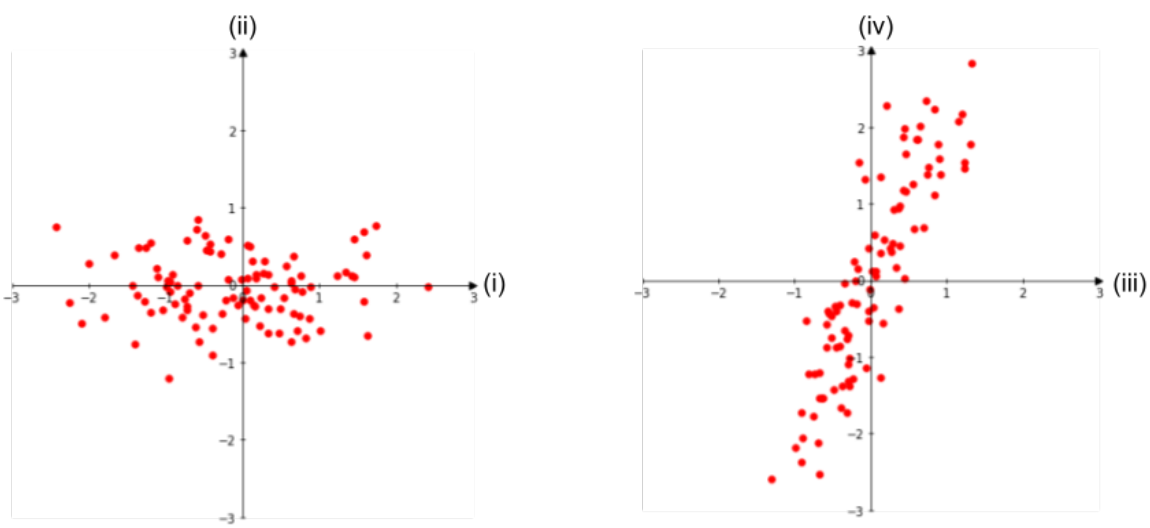


Figure 4: Projected datasets.

Solution: The concepts needed to solve this problem were explored in Note 14 and Homework 12 Q5, Q6, and Q7.

You may recall from the notebook on neuron classification — when we project data along principal components the data is aligned to orthogonal axes where as when we projected it along random components it was not aligned to any axes. So we can deduce from this that (i) and (ii) must correspond in some order to the principal components and (iii) and (iv) must correspond to the random directions. Further observation leads us to see that axis (ii) has more spread than axis (i) and therefore must correspond to the larger singular value, i.e. to the first principal component $\vec{v}_1$.

i. $\vec{v}_2$ – the corresponding axis to $\vec{v}_1$ and also an axis for which the spread of the data is axis-aligned.
ii. $\vec{v}_1$ – This is the axis with the maximal spread of the data and therefore must correspond to the largest singular value. the single axis, across both plots, across which there is maximal spread of the data.
iii. $\vec{w}_1$ or $\vec{w}_2$ – seemingly a random projection. We don't know which one is $\vec{w}_1$ and $\vec{w}_2$ since they are random unit vectors and as such are independent of the data, so we can't tell from the plot.
iv. $\vec{w}_2$ or $\vec{w}_1$ – seemingly a random projection. We don't know which one is $\vec{w}_1$ and $\vec{w}_2$ since they are random unit vectors and as such are independent of the data, so we can't tell from the plot.

(b) In order to reduce the dimension of the data, we would like to project the data onto the first k principal components along the rows of A, where k is less than the original data dimension d. **Show how to find the new coordinates $\vec{z}_i$ of the data point $\vec{x}_i$ after this projection.** You may use the SVD of A.

Solution: The concepts needed to solve this problem were explored in Note 14 and Homework 12 Q5, Q6, and Q7.
Let

$$V_k = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \tag{14}$$

Since we are projecting onto the columns of V_k, the new coordinate of $\vec{x}_i$ after dimension reduction is $\vec{z}_i = (V_k^\top V_k)^{-1} V_k^\top \vec{x}_i = V_k^\top \vec{x}_i$. Note that $V_k^\top V_k = I_k$ since V_k has orthonormal columns.
Note also that this is equivalent to saying $\vec{z}_i$ is obtained by taking the first k entries of $V^\top \vec{x}_i$.

(c) Using the data you have, you trained a classifier $\vec{w}_\star$. For any new data point after dimension reduction $\vec{z}_{\text{new}}$, the value of $\text{sign}(\vec{w}_\star^\top \vec{z}_{\text{new}})$ tells you whether the data point belongs to the "$+1$" class or to the "-1" class. Now suppose you have obtained two new data points, $\vec{z}_a$ and $\vec{z}_b$. Based on Figure 5 showing $\vec{w}_\star$, $\vec{z}_a$ and $\vec{z}_b$, **predict the class of $\vec{z}_a$ and $\vec{z}_b$ using $\vec{w}_\star$, and justify your answer.**

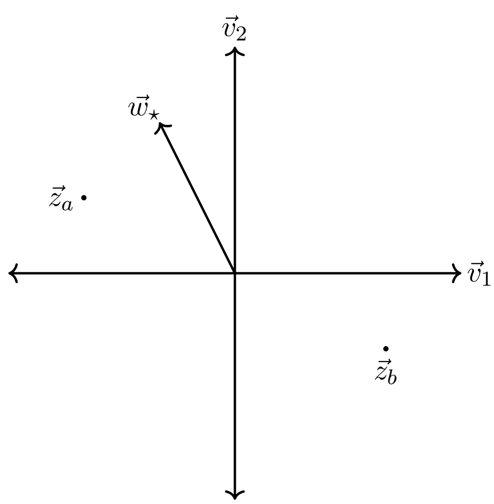


Figure 5: Dataset projected onto $\vec{v}_1$ and $\vec{v}_2$ with $\vec{w}_\star$

Solution: The concepts needed to solve this problem were explored in Note 14, Note 16, and Homework 12 Q5, Q6, and Q7.

Since we are classifying based on $\text{sign}(\vec{w}_\star^\top \vec{z}_{\text{new}})$, from the graph we can see that $\vec{w}_\star^\top \vec{z}_a > 0$, thus $\vec{z}_a$ is predicted to be in class "+1". Similarly, $\vec{w}_\star^\top \vec{z}_b < 0$, and thus $\vec{z}_b$ is predicted to be in class "−1".

(d) Assume $d = 6$, $k = 4$, and $\vec{w}_\star = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\top$. Let $A = U\Sigma V^\top$ for A defined in eq. (13), and you find that V is given by the identity matrix, i.e. $V = I_d$. Now suppose the data point for this year's big game $\vec{x}_{2021} = \begin{bmatrix} 3 & 6 & 4 & 1 & 9 & 6 \end{bmatrix}^\top$. **Would you bet on Cal or Stanford to win? Justify your answer.** A quick reminder that "+1" denotes victories of Cal and "−1" denotes victories of Stanford. *A correct guess will yield* 0 *points.*

Hint: Don't forget to project your data onto the principal components.

Solution: The concepts needed to solve this problem were explored in Note 14, Note 16, and Homework 12 Q5, Q6, and Q7.

First, we need to preprocess this data point and project it onto the k-dimensional subspace just like what we did to the training points

$$\begin{aligned}
\vec{z}_{2021} &= V_k^\top \vec{x}_{2021} \\
&= V_k^\top \begin{bmatrix} 3 \\ 6 \\ 4 \\ 1 \\ 9 \\ 6 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 6 \\ 4 \\ 1 \end{bmatrix}
\end{aligned}$$

Then, we compute the classifier's predicted value

$$\begin{aligned}
p_{2021} &= \vec{w}_\star^\top \vec{z}_{2021} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \\ 1 \end{bmatrix} \\
&= 6
\end{aligned}$$

Therefore, the classifier predicts the label for this data point to be "+1", thus you should bet for Cal to win this year!

5. Cruise Control

Suppose that we're working with a more advanced version of the robot car we built in the lab. Its state at timestep k is n dimensional, captured in $\vec{x}[k] \in \mathbb{R}^n$. The control at each timestep $\vec{u}[k] \in \mathbb{R}^m$. The system evolves according to the discrete-time equation

$$\vec{x}[k+1] = A\vec{x}[k] + B\vec{u}[k]. \tag{15}$$

We know the values of the $n \times n$ matrix A and the $n \times m$ matrix B (say for example estimated through system identification). For all parts, the initial condition is $\vec{x}[0] = \vec{0}$.

(a) We want to transform our system to a nicer set of coordinates in the S basis. S is an $n \times n$ invertible matrix. Let us write the transformed state as $\vec{z}[k] = S^{-1}\vec{x}[k]$ for all k. **Show that eq. (15) can be written in the form**

$$\vec{z}[k+1] = \widetilde{A}\vec{z}[k] + \widetilde{B}\vec{u}[k]. \tag{16}$$

with $\widetilde{A} = S^{-1}AS$ and $\widetilde{B} = S^{-1}B$. Show your work.

Solution: The concepts needed to solve this problem were explored in Note 3A and Discussion 4B.

$$\vec{x}[k+1] = A\vec{x}[k] + B\vec{u}[k] \tag{17}$$
$$S\vec{z}[k+1] = AS\vec{z}[k] + B\vec{u}[k] \tag{18}$$
$$\vec{z}[k+1] = S^{-1}AS\vec{z}[k] + S^{-1}B\vec{u}[k] \tag{19}$$

(b) **Prove that the system in eq. (16) is controllable if and only if the system in eq. (15) is controllable. Show your work.**

(*Hint: Connect the controllability matrix of the system in eq.* (16) *to the controllability matrix of the system in eq.* (15).)

Solution: The concepts needed to solve this problem were explored in Note 9 and Homework 9 Q2. We have

$$C_{\vec{z}} = \begin{bmatrix} \widetilde{B} & \widetilde{A}\widetilde{B} & \cdots & \widetilde{A}^{n-1}\widetilde{B} \end{bmatrix} \tag{20}$$

Note that

$$A^t = \left(S\widetilde{A}S^{-1}\right)^t = S\widetilde{A}S^{-1}S\widetilde{A}S^{-1}\left(S\widetilde{A}S^{-1}\right)^{t-2} = S\widetilde{A}^2S^{-1}\left(S\widetilde{A}S^{-1}\right)^{t-2} = \cdots \tag{21}$$
$$= S\widetilde{A}^tS^{-1}. \tag{22}$$

So now we can write the controllability matrix for eq. (15):

$$C_{\vec{x}} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{23}$$
$$= \begin{bmatrix} B & S\widetilde{A}S^{-1}B & \cdots & S\widetilde{A}^{n-1}S^{-1}B \end{bmatrix} \tag{24}$$
$$= \begin{bmatrix} SS^{-1}B & S\widetilde{A}S^{-1}B & \cdots & S\widetilde{A}^{n-1}S^{-1}B \end{bmatrix} \tag{25}$$
$$= S\begin{bmatrix} S^{-1}B & \widetilde{A}S^{-1}B & \cdots & \widetilde{A}^{n-1}S^{-1}B \end{bmatrix} \tag{26}$$

$$= S\begin{bmatrix}\widetilde{B} & \widetilde{A}\widetilde{B} & \cdots & \widetilde{A}^{n-1}\widetilde{B}\end{bmatrix} \quad (27)$$
$$= SC_{\vec{z}}. \quad (28)$$

Since S is a basis matrix, $\text{rank}(C_{\vec{x}}) = \text{rank}(C_{\vec{z}})$, and so $\text{rank}(C_{\vec{x}}) = n$ if and only if $\text{rank}(C_{\vec{z}}) = n$.

(c) Suppose (just for this problem subpart) that the system in (15) is controllable, and define its controllability matrix as $C \in \mathbb{R}^{n\times mn}$. We want to reach a goal state $\vec{g} \in \mathbb{R}^n$ in exactly n timesteps; that is, we want $\vec{x}[n] = \vec{g}$. Recall $\vec{x}[0] = \vec{0}$.

We define the sequence of minimum energy controls as $\vec{u}^\star = \begin{bmatrix}\vec{u}^\star[n-1] \\ \vdots \\ \vec{u}^\star[0]\end{bmatrix}$ where

$$\vec{u}^\star = \underset{\vec{u}}{\text{argmin}}\|\vec{u}\|^2 \quad (29)$$
$$\text{s.t. } C\vec{u} = \vec{g}. \quad (30)$$

Prove that $\vec{u}^*$ is orthogonal to the nullspace of C.

(*Hint: Consider a solution of $C\vec{u} = \vec{g}$ as $\vec{u}_{\text{sol}} = \vec{u}_{\text{null}} + \vec{u}_{\text{other}}$, where $\vec{u}_{\text{null}}$ is the component of $\vec{u}_{\text{sol}}$ in the nullspace of C, (i.e. $\vec{u}_{\text{null}}$ the projection of $\vec{u}_{\text{sol}}$ onto the nullspace of C).*)

While you have seen this proof in lecture/HW/notes, we are asking you to redo it from scratch here, just stating that it was done in class will receive no credit.

Solution: The concepts needed to solve this problem were explored in Note 12, Lecture 10B, and Homework 11 Q4.

Let us suppose that we found some $\vec{u}_{\text{sol}}$ that satisfies (30). We decompose $\vec{u}_{\text{sol}}$ into a component $\vec{u}_{\text{null}} \in \text{Null}(C)$ and a component $\vec{u}_{\text{other}} \perp \text{Null}(C)$ such that

$$\vec{u}_{\text{sol}} = \vec{u}_{\text{null}} + \vec{u}_{\text{other}}.$$

Plugging this into (30) gives us

$$\begin{aligned}\vec{g} &= C\vec{u}_{\text{sol}} \\ &= C(\vec{u}_{\text{null}} + \vec{u}_{\text{other}}) \\ &= C\vec{u}_{\text{null}} + C\vec{u}_{\text{other}} \\ &= C\vec{u}_{\text{other}}.\end{aligned}$$

The implication above tells us that $\vec{u}_{\text{other}}$ also satisfies (30).

Now we can consider the norm squared of $\vec{u}_{\text{sol}}$

$$\begin{aligned}\|\vec{u}_{\text{sol}}\|^2 &= \vec{u}_{\text{sol}}^\top\vec{u}_{\text{sol}} \\ &= (\vec{u}_{\text{null}} + \vec{u}_{\text{other}})^\top(\vec{u}_{\text{null}} + \vec{u}_{\text{other}}) \\ &= \vec{u}_{\text{null}}^\top\vec{u}_{\text{null}} + \vec{u}_{\text{null}}^\top\vec{u}_{\text{other}} + \vec{u}_{\text{null}}^\top\vec{u}_{\text{other}} + \vec{u}_{\text{other}}^\top\vec{u}_{\text{other}}.\end{aligned}$$

Recall that we $\vec{u}_{\text{null}}$ is orthogonal to $\vec{u}_{\text{other}}$ by definition, meaning that $\vec{u}_{\text{null}}^\top\vec{u}_{\text{other}} = 0$. So, we can clean up the expression above as

$$\|\vec{u}_{\text{sol}}\|^2 = \vec{u}_{\text{null}}^\top \vec{u}_{\text{null}} + \vec{u}_{\text{other}}^\top \vec{u}_{\text{other}}$$
$$\implies \|\vec{u}_{\text{sol}}\|^2 = \|\vec{u}_{\text{null}}\|^2 + \|\vec{u}_{\text{other}}\|^2$$

which is essentially the Pythagorean Theorem. If we are trying to minimize $\|\vec{u}_{\text{sol}}\|^2$ then we should set $\vec{u}_{\text{null}} = \vec{0}$ since $\vec{u}_{\text{other}}$ already solves (30). Thus, our final optimal solution is $\vec{u^*} = \vec{u}_{\text{other}}$, which is completely orthogonal to the nullspace of C.

(d) Now let us work in the standard basis, with the system in eq. (15). Suppose $n = 3$ and $m = 1$ (so that $A \in \mathbb{R}^{3\times 3}$, $B \in \mathbb{R}^3$, $\vec{x}[k] \in \mathbb{R}^3$, and $u[k] \in \mathbb{R}$). The SVD of the controllability matrix C is given as

$$C = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vec{v}_3^\top \end{bmatrix}, \tag{31}$$

with $\alpha > \beta > 0$.

Is the system controllable? Justify your answer.

If the system *is* controllable, find a sequence of inputs $\vec{u} = \begin{bmatrix} u[2] & u[1] & u[0] \end{bmatrix}^\top$, such that $\vec{x}[3] = \vec{g}$, for a specific $\vec{g} \in \mathbb{R}^3$. (Here $\vec{u}$ should be a function of $\vec{g}$).

If the system *is not* controllable, find a $\vec{g} \in \mathbb{R}^3$ that is unreachable by the system, i.e. find $\vec{g}$ such that there is *no* sequence of inputs $\vec{u}$ that makes $\vec{x}[3] = \vec{g}$.

All answers for this problem part should be in terms of $\vec{w}_i$, $\vec{v}_i$, α, and β.

(*Hint: Remember how the SVD is connected to the column space and null space of the matrix and that $\vec{x}[0] = \vec{0}$.*)

Solution: The concepts needed to solve this problem were explored in Note 9, Discussion 8B Q3, Note 13, and Discussion 12A Q1.

The system is not controllable; we have $\text{rank}(C) = 2 < 3 = n$, since C has one zero singular value. Since the initial condition $\vec{x}[0] = \vec{0}$, we know that

$$\vec{x}[3] = A^3\vec{x}[0] + C\vec{u} = C\vec{u}. \tag{32}$$

Thus, the vectors that are reachable are exactly $\text{col}(C)$. Since from the properties of SVD we know that $\text{col}(C) = \text{span}(\vec{w}_1, \vec{w}_2)$ and $\vec{w}_3 \perp \text{span}(\vec{w}_1, \vec{w}_2)$, we know that $\vec{w}_3$ is unreachable.

(e) We continue the setup of the previous part, repeated here. We work in the standard basis, with the system in eq. (15). The SVD of the controllability matrix C is given as in (31), with $\alpha > \beta > 0$.

Let $H \subseteq \mathbb{R}^3$ be the vector subspace of inputs $\vec{u} = \begin{bmatrix} u[2] & u[1] & u[0] \end{bmatrix}^\top$ which set $\vec{x}[3] = \vec{0}$. **Give a basis for H.** Justify your answer.

All answers for this problem part should be in terms of $\vec{w}_i$, $\vec{v}_i$, α, and β. Show your work.

(*Hint: Remember that $\vec{x}[0] = \vec{0}$ and $\vec{x}[3] = C\vec{u}$.*)

Solution: The concepts needed to solve this problem were explored in Note 9, Discussion 8B Q3, Note 13, and Discussion 12A Q1.

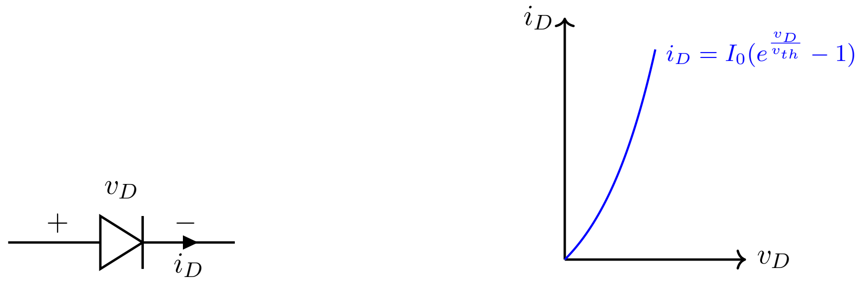
Again, we know that

$$\vec{x}[3] = C\vec{u}. \tag{33}$$

Then $\vec{x}[3] = \vec{0}$ if and only if $\vec{u} \in \operatorname{null}(C) = \operatorname{span}(\vec{v}_3)$ — since this is the part of V that corresponds to the 0 singular value. Thus $\vec{v}_3$ will be the basis for H.

6. Nonlinear Circuit Analysis and Control

So far, we have mainly focused on analyzing circuits with linear circuit elements, including resistors, capacitors, and inductors. However, we now have the tools to analyze circuits with nonlinear components. One such component is the diode. Diodes show up in many circuit applications, such as a buck-boost converter, which is a DC-to-DC converter commonly used to raise or lower some supply voltage and feed it to some other part of your circuit. We give a circuit diagram of a diode as well as its defining IV relationship below.



(a) Diode circuit diagram

(b) $i_D = I_0(e^{\frac{v_D}{v_{th}}} - 1)$

Figure 6: Diode circuit element description

For simplicity, we will be assuming parameters (perhaps unrealistically) such that the I-V relationship for our diode is:

$$i_D = e^{v_D} - 1. \tag{34}$$

(a) We want to analyze the circuit below.

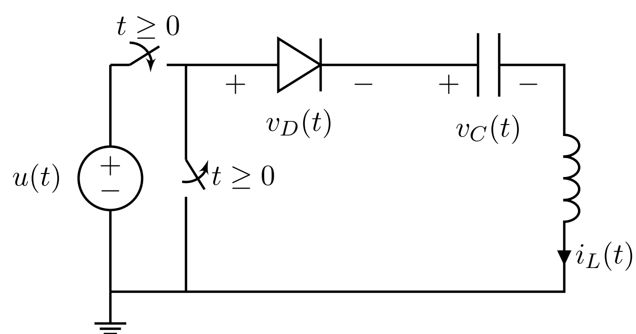


Figure 7: Diode LC Circuit Diagram

First, we'll define a model where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$.

Use KCL, KVL, and the element I-V relationships to get a system of differential equations that describe $\vec{x}(t)$ for $t \geq 0$ as a vector-valued function in terms of $v_C(t), i_L(t), u(t)$:

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(v_C, i_L, u) = \begin{bmatrix} f_1(v_C, i_L, u) \\ f_2(v_C, i_L, u) \end{bmatrix}.$$

What are f_1 and f_2? Note that these may be non-linear functions, but they cannot contain derivatives. Show your work.

Solution: The concepts needed to solve this problem were explored in Homework 2 Q4 and EECS 16A.

KCL at the node between the capacitor and inductor gives:

$$C\frac{d}{dt}v_C(t) = i_L(t) \implies \frac{d}{dt}v_C(t) = \frac{1}{C}i_L(t) = f_1$$

KVL gives:

$$\begin{aligned} u(t) &= v_D(t) + v_C(t) + L\frac{d}{dt}i_L(t) \\ \implies u(t) &= \ln\big(i_L(t)+1\big) + v_C(t) + L\frac{d}{dt}i_L(t) \\ \implies \frac{d}{dt}i_L(t) &= -\frac{1}{L}v_C(t) - \frac{1}{L}\ln\big(i_L(t)+1\big) + \frac{1}{L}u(t) = f_2 \end{aligned}$$

Putting everything together, we get

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{C}i_L(t) \\ -\frac{1}{L}v_C(t) - \frac{1}{L}\ln\big(i_L(t)+1\big) + \frac{1}{L}u(t) \end{bmatrix}$$

(b) Say that one of the equations you got above was in the form:

$$\frac{d}{dt}y(t) = \frac{1}{L}\ln\big(y(t)+a\big) + \frac{1}{L}u(t), \tag{35}$$

where $a \in \mathbb{R}$ is a constant and $u(t)$ can be thought of as a control input. (This is not necessarily the correct answer for the earlier part). You choose $y^* = 0$ and $u^* = 1\,\text{V}$ as the operating point. **Linearize the above equation (35) about this operating point.** Recall that $\frac{d}{dz}\ln(z) = \frac{1}{z}$. Show your work.

Solution: The concepts needed to solve this problem were explored in Note 15 and Homework 13 Q2.

We have the system

$$\frac{d}{dt}y(t) = f\left(y(t), u(t)\right).$$

The function can be linearized around $y^* = 0$ and $u^* = 1\,\text{V}$ as follows:

$$\begin{aligned} f\left(y(t),u(t)\right) &\approx f(y^*,u^*) + \left(\frac{\partial f}{\partial y}\Big|_{y(t)=y^*}\right)\left(y(t)-y^*\right) + \left(\frac{\partial f}{\partial u}\Big|_{u(t)=u^*}\right)\left(u(t)-u^*\right) \\ &= \frac{1}{L}\ln(0+a) + \frac{1}{L}\cdot 1 + \frac{1}{L}\left(\frac{1}{y(t)+a}\Big|_{y(t)=0}\right)\left(y(t)-0\right) + \frac{1}{L}\left(u(t)-1\right) \\ &= \frac{1}{aL}y(t) + \frac{1}{L}u(t) + \frac{1}{L}\ln(a) \\ \implies \frac{d}{dt}y(t) &= \frac{1}{aL}y(t) + \frac{1}{L}u(t) + \frac{1}{L}\ln(a) \end{aligned}$$

Alternatively, you could have noticed that f is already linear with respect to $u(t)$ and so you only need to linearize the $\ln$ term.

(c) Now suppose you chose a capacitance and inductance such that the linearized model for the system in

Fig. 7 around a particular equilibrium point looked like:

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}}_{A}\vec{x}(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(t) \tag{36}$$

In order to solve this system, you need to convert A into a more convenient form.

Find an orthonormal matrix V and an upper-triangular matrix T such that $A = VTV^\top$. Show your work.

Hint: You may use the fact that the eigenvalues of A are $-2, -2$, with eigenspace $\text{span}(\vec{v}_1)$, where $\vec{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$.

Solution: The concepts needed to solve this problem were explored in Note 11, Homework 10 Q5, and Discussion 10A.

From the algorithm discussed in lecture, we can construct an orthonormal basis recursively, starting with the single eigenvector given to us. We first want to find some vector $\vec{r}_1$ that is orthonormal to $\vec{v}_1$, which can be done either from inspection or Gram-Schmidt.

If you use Gram-Schmidt, then there will be 2 cases as you will orthonormalize $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Case 1:

$$\begin{aligned}
\vec{q}_1 &= \vec{e}_1 - \langle \vec{v}_1, \vec{e}_1 \rangle \vec{v}_1 \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{5}} \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix} \\
\implies \vec{r}_1 &= \frac{\vec{q}_1}{\|\vec{q}_1\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}
\end{aligned}$$

Case 2:

$$\begin{aligned}
\vec{q}_1 &= \vec{e}_2 - \langle \vec{v}_1, \vec{e}_2 \rangle \vec{v}_1 \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{5}} \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \end{bmatrix} \\
\implies \vec{r}_1 &= \frac{\vec{q}_1}{\|\vec{q}_1\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}
\end{aligned}$$

Then from lecture we know the V basis should upper-triangularize a 2×2 matrix:

$$V = \begin{bmatrix} \vec{v}_1 & \vec{r}_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\begin{aligned} T = V^\top A V &= \begin{bmatrix} \vec{v}_1^\top A \vec{v}_1 & \vec{v}_1^\top A \vec{r}_1 \\ \vec{r}_1^\top A \vec{v}_1 & \vec{r}_1^\top A \vec{r}_1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \vec{v}_1^\top \vec{v}_1 & \vec{v}_1^\top A \vec{r}_1 \\ \lambda_1 \vec{r}_1^\top \vec{v}_1 & \vec{r}_1^\top A \vec{r}_1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -5 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

Alternatively, if we use $\vec{r}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}$

$$V = \begin{bmatrix} \vec{v}_1 & \vec{r}_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\begin{aligned} T = V^\top A V &= \begin{bmatrix} \vec{v}_1^\top A \vec{v}_1 & \vec{v}_1^\top A \vec{r}_1 \\ \vec{r}_1^\top A \vec{v}_1 & \vec{r}_1^\top A \vec{r}_1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \vec{v}_1^\top \vec{v}_1 & \vec{v}_1^\top A \vec{r}_1 \\ \lambda_1 \vec{r}_1^\top \vec{v}_1 & \vec{r}_1^\top A \vec{r}_1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 5 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

(d) We now want to move the eigenvalues of our linearized system more left in the complex plane to have our state approach the equilibrium point faster. The system is given below again for convenience:

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 4 \end{bmatrix}}_{\vec{b}} u(t).$$

Design a state-feedback controller $u = \vec{k}^\top \vec{x} = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}$ to move the eigenvalues of the system to $\lambda = -4, -5$. That is, find k_1, k_2 to give the desired eigenvalues.

Solution: The concepts needed to solve this problem were explored in Note 8 and Homework 9 Q2. If we set $\delta u = \vec{k}^\top \delta \vec{x} = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x$, then the closed loop system becomes

$$\frac{d}{dt}\delta\vec{x} = (A + \vec{b}\vec{k}^\top)\delta\vec{x}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4k_1 & 4k_2 \end{bmatrix} \right) \delta\vec{x}$$
$$= \begin{bmatrix} 0 & 1 \\ -4 + 4k_1 & -4 + 4k_2 \end{bmatrix} \delta\vec{x}$$

We want the eigenvalues to be $-4, -5$ so the desired characteristic polynomial is $(\lambda + 4)(\lambda + 5) = \lambda^2 + 9\lambda + 20$. The characteristic polynomial of our closed loop matrix is

$$\begin{aligned} -\lambda(-4 + 4k_2 - \lambda) - (-4 + 4k_1) &= \lambda^2 + (4 - 4k_2)\lambda + (4 - 4k_1) \\ &= \lambda^2 + 9\lambda + 20 \end{aligned}$$

Thus, we need $k_1 = -4$, and $k_2 = \frac{-5}{4}$ so $\vec{k}^\top = \begin{bmatrix} -4 & \frac{-5}{4} \end{bmatrix}$.

