EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet

Discussion 15B

In this discussion we review select questions from previous years.

(a) **Slot 1 : 11am - 12pm**

- 204 Wheeler (Krishna) : **Q1, Q3**
- 222 Wheeler (Kunmo) : **Q5, Q6**
- 241 Cory (Manav) : **Q2, Q4**

(b) **Slot 2 : 12pm - 1pm**

- 103 Moffitt (Divija) : **Q2, Q4** (extended section : 12pm-2pm)
- 108 Wheeler (Kunmo) : **Q5, Q6**
- 219 Dwinelle (Neelesh) : **Q1, Q3**
- 3109 Etcheverry (Manav) : **Q2, Q4**

(c) **Slot 3 : 2pm - 3pm**

- 108 Wheeler (Maxwell) : **Q1, Q3** (extended section : 2pm-4pm)
- 20 Wheeler (Ashwin) : **Q5, Q6**
- 3111 Etcheverry (Manav) : **Q2, Q4**

(d) **Slot 4 : 5pm - 6pm**

- 170 SOCS (Maxwell) : **Q1, Q3**
- 106 Moffitt (Kunmo) : **Q5, Q6**

1. Separation of Variables and Uniqueness

Recall that the classic scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{1}$$

with initial condition $x(0) = x_0 \neq 0$ has the unique solution $x(T) = x_0 e^{\lambda T}$ for all $T \geq 0$.

(Note: to avoid variable-name confusion here, we are using T as the argument of the solution $x(T)$.)

The separation of variables approach to getting a guess for this problem would proceed as follows:

$$\begin{align}
\frac{d}{dt}x(t) &= \lambda x(t) \tag{2}\\
\frac{dx}{dt} &= \lambda x \tag{3}\\
\frac{dx}{x} &= \lambda dt \text{ separating variables to sides} \tag{4}\\
\int_{x_0}^{x(T)} \frac{dx}{x} &= \int_0^T \lambda dt \text{ integrating both sides from where they start to where they end up} \tag{5}\\
\ln\big(x(T)\big) - \ln(x_0) &= \lambda T \tag{6}\\
\ln\big(x(T)\big) &= \ln(x_0) + \lambda T \tag{7}\\
x(T) &= x_0 e^{\lambda T} \text{ exponentiating both sides} \tag{8}
\end{align}$$

and in this case it gave a good guess. Of course, this guess needed to be justified by a uniqueness proof, which you did in the homework.

This exam problem asks you to carry out this program for the time-varying differential equation:

$$\frac{d}{dt}x(t) = \lambda(t)x(t) \tag{9}$$

with initial condition $x(0) = x_0 \neq 0$. You can assume that $\lambda(t)$ is a nice continuously differentiable function of time t that is bounded.

(a) (8 pts) Use the separation of variables approach to get a guess for the solution to the differential equation (9) — namely $\frac{d}{dt}x(t) = \lambda(t)x(t)$ — with initial condition $x(0) = x_0 \neq 0$. **Show work and give a formula for $x(T)$ for $T \geq 0$.**

(HINT: It is fine if your answer involves a definite integral.)

(If you can't solve this for a general $\lambda(t)$, for partial credit, feel free to just consider the special case of $\lambda(t) = -2 - \sin(t)$ and give a guess for that case.)

(You can also get full credit if you follow the approach from discussion section of taking a piecewise-constant approximation and then taking a limit, but that might involve more work.)

Solution:

$$\frac{d}{dt}x(t) = \lambda(t)x(t) \tag{10}$$
$$\frac{dx}{dt} = \lambda(t)x \tag{11}$$
$$\frac{dx}{x} = \lambda(t)dt \text{ separating variables to sides} \tag{12}$$
$$\int_{x_0}^{x(T)} \frac{dx}{x} = \int_0^T \lambda(t)dt \text{ integrating both sides from where they start to where they end up} \tag{13}$$
$$\ln\big(x(T)\big) - \ln(x_0) = \int_0^T \lambda(t)dt \tag{14}$$
$$\ln\big(x(T)\big) = \ln(x_0) + \int_0^T \lambda(t)dt \tag{15}$$
$$x(T) = x_0 e^{\int_0^T \lambda(t)dt} \text{ exponentiating both sides and folding constants} \tag{16}$$

which is a pretty reasonable guess since it definitely agrees with the correct answer if $\lambda(t)$ was just a constant.

For the special case of $\lambda(t) = -2 - \sin(t)$, we know that $\int_0^T \lambda(t)dt = \int_0^T -2 - \sin(t)dt = -2T + \cos(T) - 1$ and so $x(T) = x_0 e^{-2T+\cos(T)-1}$ is our guessed solution.

It turns out that the same limiting argument invoking piecewise-constants and Reimann sums that was done in discussion would also have resulted in the same guess.

(b) (12 pts) **Prove the uniqueness of the solution — i.e. that if any function solves differential equation (9) — namely $\frac{d}{dt}x(t) = \lambda(t)x(t)$ — with the given initial condition $x(0) = x_0 \neq 0$, then it must in fact be the same as your guessed solution everywhere for $T \geq 0$.**

(HINT: A ratio-based argument might be useful. You don't actually need to know the exact form of your guessed solution to carry out much of this argument, but you do need the fact that it is never zero and that it solves (9).)

Solution: First, we need to know how our guessed solution behaves.

Plugging in the intial condition into $x(T) = x_0 e^{\int_0^T \lambda(t)dt}$, we get $x(0) = x_0 e^{\int_0^0 \lambda(t)dt}$, we get $x(0) = x_0 e^0 = x_0$.

Before plugging into (9), we first write the solution using t instead of T and changing the dummy variable for the integral to be τ: $x(t) = x_0 e^{\int_0^t \lambda(\tau)d\tau}$.

Plugging into (9), we get

$$\frac{d}{dt}x(t) = \frac{d}{dt}x_0 e^{\int_0^t \lambda(\tau)d\tau} \tag{17}$$
$$= x_0 \lambda(t) e^{\int_0^t \lambda(\tau)d\tau} \tag{18}$$
$$= \lambda(t)x(t) \tag{19}$$

which satisfies the differential equation.

Solution: Consider any candidate solution $y(t)$ to (9) that satisfies the given initial condition $y(0) = x_0 \neq 0$. This means that $\frac{d}{dt}y(t) = \lambda(t)y(t)$ for all $t \geq 0$.

We know that our guessed solution $x(t) = x_0 e^{\int_0^t \lambda(\tau)d\tau}$ is never zero because $x_0 \neq 0$ and the finite integral of a bounded function cannot be $-\infty$. This means that we are free to consider the ratio $z(t) = \frac{y(t)}{x(t)}$. We know that $z(0) = \frac{y(0)}{x(0)} = \frac{x_0}{x_0} = 1$.

Looking at the derivative of $z(t)$, we have:

$$\frac{d}{dt} z(t) = \frac{d}{dt} \frac{y(t)}{x(t)} \tag{20}$$

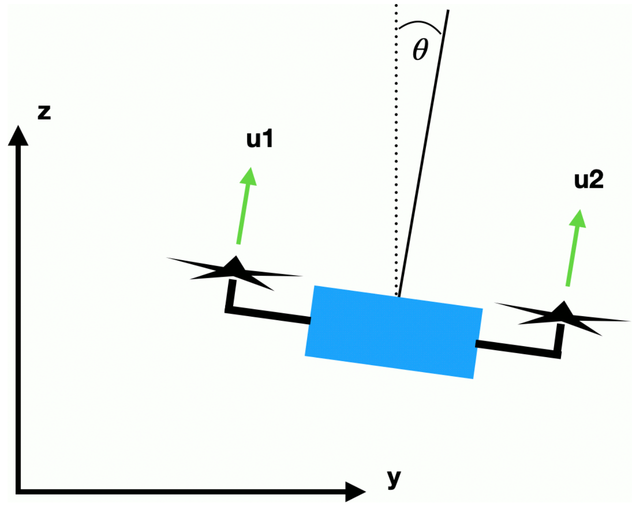
$$= \frac{(\frac{d}{dt} y(t)) x(t) - y(t) \frac{d}{dt} x(t)}{(x(t))^2} \tag{21}$$

$$= \frac{\lambda(t) y(t) x(t) - y(t) \lambda(t) x(t)}{(x(t))^2} \tag{22}$$

$$= 0. \tag{23}$$

Since $\frac{d}{dt} z(t) = 0$ for all $t \geq 0$, it is not changing, and is therefore a constant. This means that $z(t) = z(0) = 1$ which implies that $\frac{y(t)}{x(t)} = 1$ which means that $y(t) = x(t)$, thereby establishing uniqueness for our guessed solution.

2. Controlling a Quadrotor to Hover



In this problem you will design a controller which will make a planar quadrotor hover. The quadrotor we will consider is defined by the following state space model:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{v_y}(t) \\ \dot{\theta}(t) \\ \dot{\omega}(t) \\ \dot{z}(t) \\ \dot{v_z}(t) \end{bmatrix} = \begin{bmatrix} v_y(t) \\ \frac{sin(\theta(t))}{m}(u_1(t)+u_2(t)) \\ \omega(t) \\ \alpha(u_1(t)-u_2(t)) \\ v_z(t) \\ \frac{cos(\theta(t))}{m}(u_1(t)+u_2(t)) - g \end{bmatrix}$$

Here $y(t)$ denotes lateral position, $z(t)$ the altitude, $v_y(t)$ and $v_z(t)$ the corresponding linear velocities, $\theta(t)$ the roll angle, and $\omega(t)$ the angular velocity. The parameters α and m are positive, real constants. The controls $u_1(t)$ and $u_2(t)$ are the thrusts generated by the left and right propellors.

The thrust of each propellor can be positive or negative.

Define the vectors

$$x(t) := \begin{bmatrix} y(t) \\ v_y(t) \\ \theta(t) \\ \omega(t) \\ z(t) \\ v_z(t) \end{bmatrix}, \quad u(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

(a) An equilibrium point for this system is given by

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \\ 0 \end{bmatrix}, \quad u^* = \begin{bmatrix} \frac{mg}{2} \\ \frac{mg}{2} \end{bmatrix}$$

Here $h > 0$ is a specified altitude.

Do there exist any other equilibrium points for this system which satisfy $y^* = 0$ and $z^* = h$? If so, what are they? If not, explain why not.

Solution: Yes there exist infinite such equilibria (or two if θ is modulated by 2π.) The equilibra are given by

$$\begin{bmatrix} y^* \\ v_y^* \\ \theta^* \\ \omega^* \\ z^* \\ v_z^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k\pi \\ 0 \\ h \\ 0 \end{bmatrix}, \quad \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} = \begin{bmatrix} (-1^k)\frac{mg}{2} \\ (-1^k)\frac{mg}{2} \end{bmatrix}, \quad k \in \mathbb{Z}$$

(b) Consider a linearization of this system, formed by taking the first-order taylor approximation of the system about the equililbrium point given in part (a). This linearized system is given by

$$\dot{\delta x}(t) = A\delta x(t) + B\delta u(t)$$

where $\delta x(t) = (x(t) - x^*)$, and $\delta u(t) = (u(t) - u^*)$. The matrices A and B are given by

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \beta_3 & -\beta_3 \\ 0 & 0 \\ \beta_4 & \beta_4 \end{bmatrix}$$

Find the parameters $\beta_1, \beta_2, \beta_3, \beta_4$.

Solution:

$$\beta_1 = \frac{\partial(\frac{sin\theta}{m}(u_1+u_2))}{\partial\theta}\bigg|_{\theta=\theta^*,u_1=u_1^*,u_2=u_2^*} = \frac{cos\theta^*}{m}(u_1^*+u_2^*) = g$$

$$\beta_2 = \frac{\partial v_z}{\partial v_z}\bigg|_{v_z=v_z^*} = 1$$

$$\beta_3 = \frac{\partial\alpha(u_1-u_2)}{\partial u_1}\bigg|_{u_1=u_1^*,u_2=u_2^*} = \alpha$$

$$\beta_4 = \frac{\partial(\frac{cos\theta}{m}(u_1+u_2)-g)}{\partial u_1}\bigg|_{\theta=\theta^*,u_1=u_1^*,u_2=u_2^*} = \frac{cos\theta^*}{m} = \frac{1}{m}.$$

(c) Is the linearized system found in part (b) stable? Explain your answer. Hint: notice that A is an upper-triangular matrix.

Solution: The system is marginally stable, since all 6 eigenvalues are 0. This is clearly seen by the fact that the eigenvalues of an upper-triangular matrix appear along the main diagonal of the matrix. Because this system is marginally stable, it is unstable.

(d) Does the matrix B have full column-rank? Explain your answer. Here you can use the fact $\beta_3 \neq 0$ and $\beta_4 \neq 0$. **Solution:** Yes. Let $b_i(k)$ denote the kth element of the ith column of B. The scalar γ which solves $\gamma b_1(4) = b_2(4)$ is $\gamma = -1$. But $\gamma b_1(6) \neq b_2(6)$. This implies b_1 and b_2 are linearly independent, and B is full column-rank.

(e) Consider the matrix $C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. Is C full row-rank? What does this imply about the ability or inability to choose arbitrary closed-loop eigenvalues for this system through use of feedback control? Explain your answer.

Solution:

$$C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & \beta_1\beta_3 & -\beta_1\beta_3 \\ 0 & 0 & 0 & 0 & \beta_1\beta_3 & -\beta_1\beta_3 & 0 & 0 \\ 0 & 0 & \beta_3 & -\beta_3 & 0 & 0 & 0 & 0 \\ \beta_3 & -\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_2\beta_4 & \beta_2\beta_4 & 0 & 0 & 0 & 0 \\ \beta_4 & \beta_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Each row is zero except in exactly one of the matrices $\{B, AB, A^2B, A^3B\}$. Each of these matrices is full rank, implying therefore that each row of C is linearly independent. Therefore C is full row-rank. Because C is full row-rank, the system (A, B) is controllable, and therefore we can design a control law $\delta u(t) := -K\delta x(t)$ such that the closed loop eigenvalues of the resultant system can be arbitrariliy chosen.

(f) Define a control law for this system of the form $\delta u(t) := -K\delta x(t)$, where K is defined as

$$K := \begin{bmatrix} 0 & 0 & 0 & \frac{k_1}{2} & 0 & \frac{k_2}{2} \\ 0 & 0 & 0 & -\frac{k_1}{2} & 0 & \frac{k_2}{2} \end{bmatrix}.$$

Find the constants k_1 and k_2 in terms of the parameters β_3 and β_4 so that two of the eigenvalues of the closed-loop system are equal to -1. **Solution:**

$$A - BK = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & \beta_1 & & & \\ & & 0 & 1 & & \\ & & & -k_1\beta_1 & 0 & \\ & & & & 0 & \beta_2 \\ & & & & & -k_2\beta_4 \end{bmatrix}$$

It can clearly be seen that if $k_1 = \frac{1}{\beta_3}$ and $k_2 = \frac{1}{\beta_4}$ the closed-loop system will have two of its diagonal entries equal to -1, implying two of the closed-loop eigenvalues will be -1.

(g) Is the closed-loop system found in part(f) stable? Explain your answer. Describe in words how the closed-loop system would respond to the initial condition

$$\delta x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}.$$

Solution: No, the system is still marginally stable, since 4 of the 6 closed-loop eigenvalues are 0. With a positive perturbation in the angular velocity (4th dimension), the closed-loop linearized system would restore the angular velocity to 0, but not before a positive angle displacement is incurred (3rd dimension). This positive angle displacement would cause the lateral velocity (2nd dimension) of the system to increase at a constant rate, which would in turn cause the lateral position (1st dimension) to increase quadratically with time. Therefore the linearized model of the quadrotor would continually accelerate away from the point $y = 0$.

3. Stability (14pts)

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

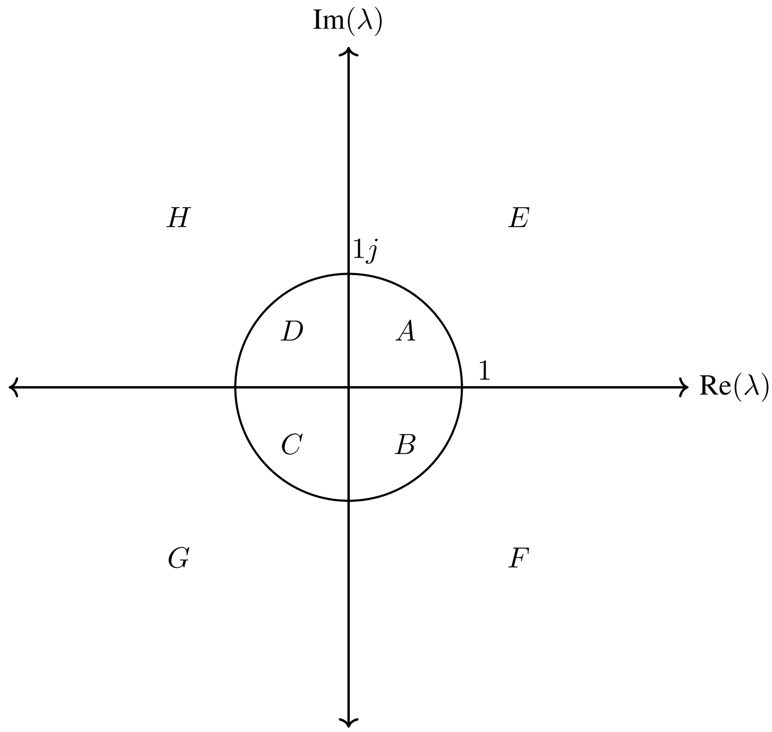


Figure 1: Complex plane divided into regions.

(a) (4pts) Consider the continuous-time system $\frac{d}{dt}x(t) = \lambda x(t) + v(t)$ and the discrete-time system $y(t+1) = \lambda y(t) + w(t)$.
In which regions can the eigenvalue λ be for a *stable* system? Fill out the table below to indicate *stable* regions. Assume that the eigenvalue λ does not fall directly on the boundary between two regions.

	A	B	C	D	E	F	G	H
Continuous Time System $x(t)$	○	○	○	○	○	○	○	○
Discrete Time System $y(t)$	○	○	○	○	○	○	○	○

Solution: For the continuous time system to be stable, we need the real part of λ to be less than zero. Hence, C,D,G,H satisfy this condition.

On the other hand, for the discrete time system to be stable, we need the norm of λ to be less than one. Hence, A,B,C,D satisfy this condition.

(b) (10pts) Consider the continuous time system

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t)$$

where λ is real and $\lambda < 0$.

Assume that $x(0) = 0$ and that $|u(t)| < \epsilon$ for all $t \geq 0$.

Prove that the solution $x(t)$ will be bounded (i.e. $\exists k$ so that $|x(t)| \leq k\epsilon$ for all time $t \geq 0$.).
(Hint: Recall that the solution to such a first-order scalar differential equation is:

$$x(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau$$

You may use this fact without proof.)

Solution: Start by taking the abolute value of both sides.

$$\begin{aligned}
|x(t)| &= \left| x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \right| && (24)\\
&= \left| \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \right| && (25)\\
&\leq \int_0^t \left| u(\tau) e^{\lambda(t-\tau)} \right| d\tau && (26)\\
&= \int_0^t \left| u(\tau) \right| \left| e^{\lambda(t-\tau)} \right| d\tau && (27)\\
&< \int_0^t \epsilon \left| e^{\lambda(t-\tau)} \right| d\tau && (28)\\
&= \epsilon \int_0^t \left| e^{\lambda(t-\tau)} \right| d\tau && (29)\\
&= \epsilon \int_0^t e^{\lambda(t-\tau)} d\tau && (30)\\
&= \epsilon \cdot \frac{e^{\lambda t} - 1}{\lambda} && (31)\\
&= \epsilon \cdot \frac{1 - e^{\lambda t} - 1}{-\lambda} && (32)\\
&\leq \epsilon \cdot \frac{1}{-\lambda}. && (33)
\end{aligned}$$

We have written things out in gory detail, you didn't need to call out each of these steps.
Hence the solution $x(t)$ will be bounded. In the above, we used the following integration:

$$\int_0^t e^{\lambda(t-\tau)} d\tau = e^{\lambda t} \cdot \int_0^t e^{-\lambda \tau} d\tau = e^{\lambda t} \cdot \left(-\frac{1}{\lambda} e^{-\lambda \tau} \right)_0^t = e^{\lambda t} \cdot \left(\frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda t} \right) = \frac{e^{\lambda t} - 1}{\lambda}.$$

4. Computing the SVD (10pts)

Consider the matrix

$$A = \begin{bmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \end{bmatrix}.$$

Write out a singular value decomposition of the matrix A in the form $U\Sigma V^T$ where U is a 2×2 orthonormal matrix, Σ is a diagonal rectangular matrix, and V is a 3×3 orthonormal matrix.

Solution: We can calculate the singular values of A by finding the eigenvalues of

$$AA^\top = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

and we see that the eigenvalues of $AA^\top$ are both 25, and hence

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}.$$

Then, we can find U by finding the set of normalized eigenvectors of $AA^\top$. Here, we see that

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Finally, to find V, we can calculate

$$U\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}.$$

Recall that $A = U\Sigma V^\top$. We observe $V^\top$ is

$$V^\top = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives us

$$V = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This completes one possible way of doing a singular value decomposition of A:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Another way to do this problem is to start from

$$A^\top A = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with corresponding eigenspace

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To find U, we can simply calculate

$$U = AV\Sigma^{-1} = \begin{bmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

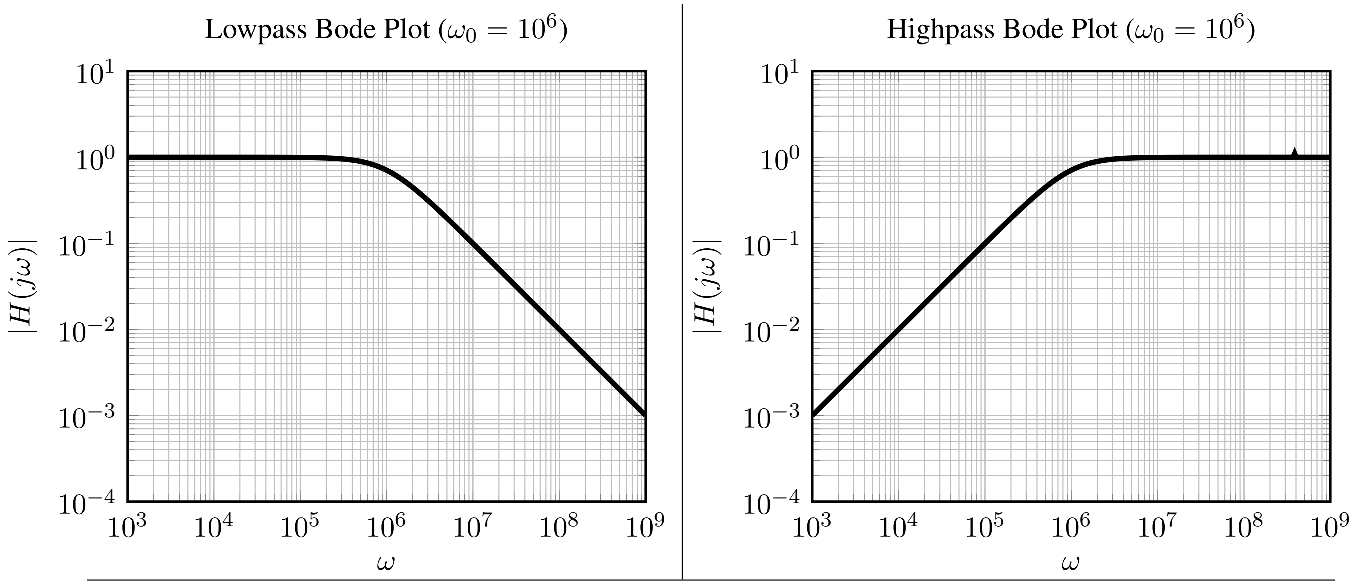
This completes another possible way of doing a singular value decomposition of A:

$$A = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There are essentially infinite possible ways of getting a singular value decomposition, and as we saw in homework, the SVD is not unique. However, your solution of the SVD must satisfy the requirements that $U^\top U = I$, $V^\top V = I$, $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$, and $A = U\Sigma V^\top$.

5. Filter Design and Bode Plots (28 pts)

On the Bode plots below, we have plotted the magnitude responses of first-order low pass filters and high pass filters using the example of cutoff frequency $\omega_0 = 10^6$.



Recall that the transfer functions for such simple low pass filters and high pass filters are:

$$H_{lowpass}(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_0}}; \qquad H_{highpass}(j\omega) = \frac{\frac{j\omega}{\omega_0}}{1 + \frac{j\omega}{\omega_0}}.$$

(a) (6 pts) We want to design a bandpass filter that can pass through a 2.4GHz WiFi signal while blocking other interfering signals — FM radio at 100MHz and WiGig at 60GHz. (Recall: Mega = 10^6 and Giga = 10^9.) We will achieve this by cascading lowpass and highpass filters, using ideal op-amp buffers in between to prevent any loading effects.

Unfortunately, when we look in the lab, we only see inductors, $1\text{k}\Omega$ resistors, and op-amps.

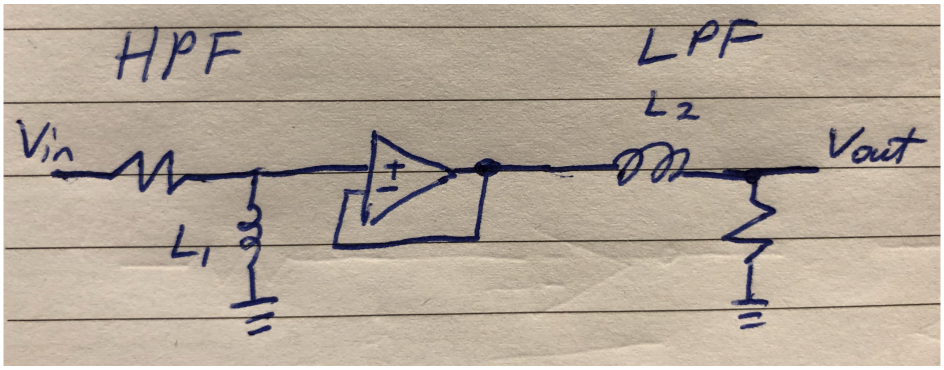
We will start by cascading a single highpass filter followed by a single lowpass filter, with an op-amp buffer in between. **Using only op-amps, two inductors, and resistors (as many as needed), draw the full band-pass filter.** Label V_{in} and V_{out} and label the two inductors with L_1 and L_2. Do not worry about picking the values for L_1 and L_2 in this part.

Solution:

The band-pass filter is drawn below.

Important notes:

- All resistors are $1k\Omega$
- An op-amp in unity-gain configuration (input to the $+$ port, output fed back to the $-$ port) separates the HPF from the LPF


HPF
LPF
L_2
V_{in}
V_{out}
L_1
+
−

(b) (8 pts) One interfering signal that we want to block is the WiGig signal at 60GHz. If we want to attenuate/reduce the magnitude of the WiGig signal by a factor of about $\sqrt{101} \approx 10$, **What is a candidate 'cutoff frequency' (in** Hz**) desired for this lowpass filter?**

What inductance value should we use for the lowpass filter? Recall that we only have resistors with $1\text{k}\Omega$ resistance. It is fine to give your inductance as a formula — you don't have to simplify it.

For your convenience, here are some calculations that may or may not be relevant:

$\frac{60\times10^9}{2\pi} = 9.549 \times 10^9$	$\frac{2.4\times10^9}{2\pi} = 382 \times 10^6$	$\frac{100\times10^6}{2\pi} = 15.9 \times 10^6$
$60 \times 10^9 \times 2\pi = 377 \times 10^9$	$2.4 \times 10^9 \times 2\pi = 15.08 \times 10^9$	$100 \times 10^6 \times 2\pi = 628 \times 10^6$

(HINT: Look at the relevant Bode plot and read off how far away in frequency from ω_0 you need to be to reduce the magnitude by the desired factor of around 10*.)*

Solution: From the Bode plot, we see that to attenuate a particular frequency by 10x, the cutoff frequency must be 10x lower.

Because we are interested in attenuating $f_{WiGig} = 60\text{GHz}$ by 10x, we find that the cutoff frequency (in Hz) must be $f_0 = \frac{1}{10} f_{WiGig} = 6\text{GHz}$.

Recall that the transfer function for a LR lowpass filter (derived using the phasor-domain voltage divider) is

$$H(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_0}} = \frac{R}{R + j\omega L} = \frac{1}{1 + \frac{j\omega}{\frac{R}{L}}}$$

This indicates that $\omega_0 = \frac{R}{L}$. Thus

$$L = \frac{R}{\omega_0} = \frac{1\text{k}\Omega}{2\pi \cdot 6 \times 10^9\text{Hz}} = \frac{1}{12\pi \times 10^6} \approx \frac{1}{377 \times 10^5} \approx 26.5\text{nH}$$

(c) (14 pts) Another interfering signal that we want to block is FM radio at 100MHz and we want to reduce its magnitude by a factor of around 100. We decide to use multiple highpass filters in a row (separated by ideal op-amp buffers) to attenuate the FM radio signal more strongly. We design the system with the highpass filter cutoff frequencies all at 1GHz. In this case, **what inductor value should each of the highpass filters use?** Recall that we only have resistors with $1\text{k}\Omega$ resistance. It is fine to give your inductance as a formula — you don't have to simplify it.

For your convenience, here are some calculations that may or may not be relevant:

$\frac{60\times10^9}{2\pi} = 9.549\times10^9$	$\frac{2.4\times10^9}{2\pi} = 382\times10^6$	$\frac{100\times10^6}{2\pi} = 15.9\times10^6$
$60\times10^9\times2\pi = 377\times10^9$	$2.4\times10^9\times2\pi = 15.08\times10^9$	$100\times10^6\times2\pi = 628\times10^6$

How many highpass filters must we cascade in order to attenuate the FM signal at 100MHz by a factor of around 100?

Draw the full circuit for your complete filter including op-amp buffers, the lowpass filter, and the highpass filters.

Solution:

We are given a cutoff frequency of $f_0 = 1\text{GHz}$. We can therefore solve for the inductor value:

$$L = \frac{R}{\omega_0} = \frac{R}{2\pi f_0} = \frac{1\text{k}\Omega}{2\pi 1\times10^9 Hz} = \frac{1}{2\pi\times10^6}\text{henry} \approx 159\text{nH}$$

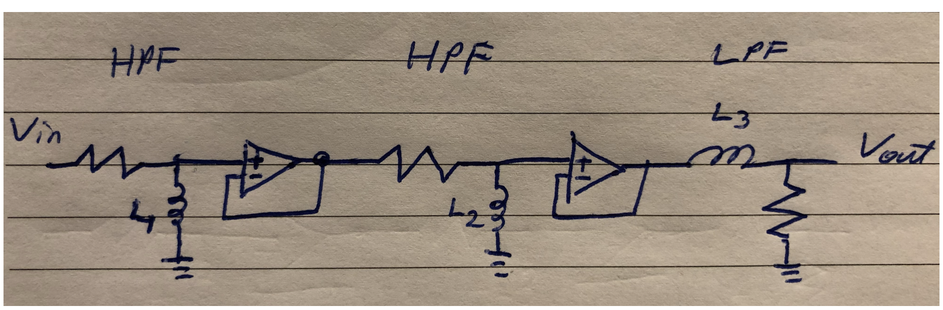
Because the filter cutoff frequency is 10x higher than the FM radio signal, the FM signal is attenuated by a factor of roughly 10x by a single filter.

Therefore, to get a factor of 100 attenuation, we must use 2 filters in cascade (each one providing 0x attenuation, for a total of 100x).

The full band-pass filter is drawn below.

Important notes:

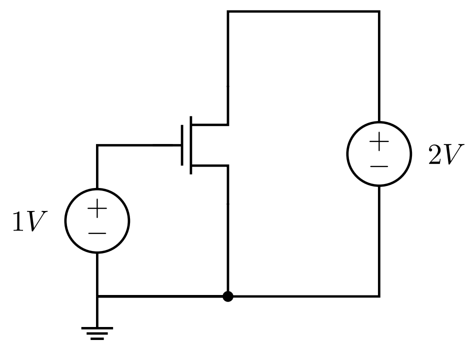
- All resistors are $1k\Omega$
- An op-amp in unity-gain configuration (input to the $+$ port, output fed back to the $-$ port) separates the HPF from the LPF
- $L_1 = L_2 = 159\text{nH}$ for the highpass filters
- $L_3 = 26.5\text{nH}$ for the lowpass filters

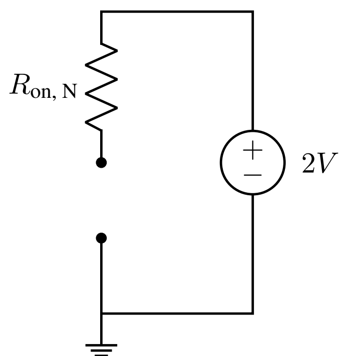


6. Transistor Behavior (12 pts)

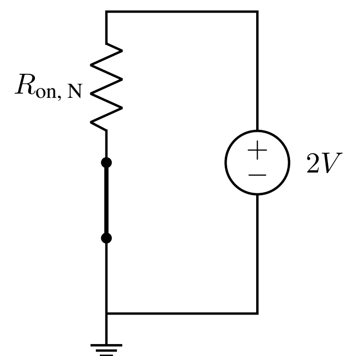
For all NMOS devices in this problem, $V_{tn} = 0.5\text{V}$. For all PMOS devices in this problem, $|V_{tp}| = 0.6\text{V}$.

(a) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**





Circuit A

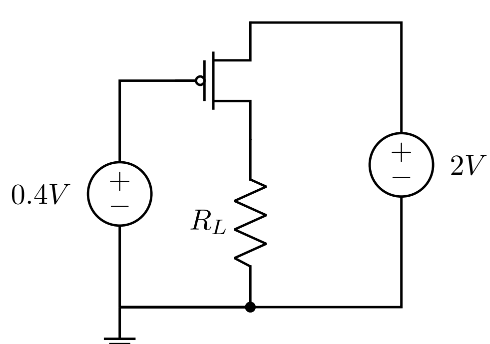


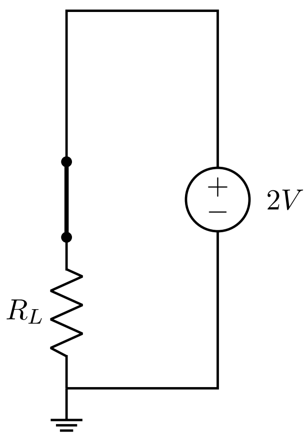
Circuit B

	A	B
Equivalent Circuit	◯	◯

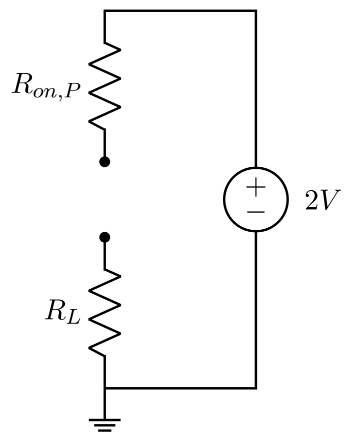
Solution: For the NMOS, $V_{GS} = 1V > V_{tn} = 0.5V$, so the NMOS transistor is on. Thus circuit B is equivalent.

(b) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**

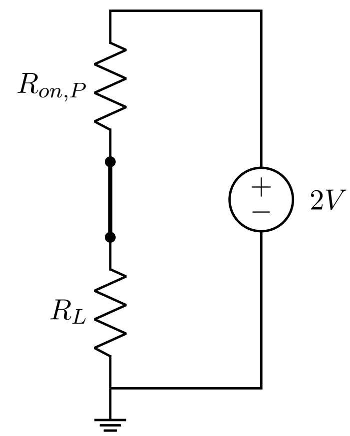




Circuit A



Circuit B

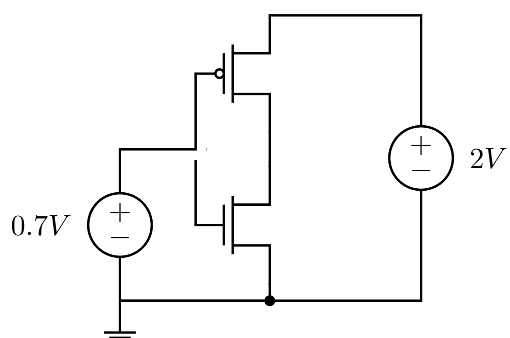


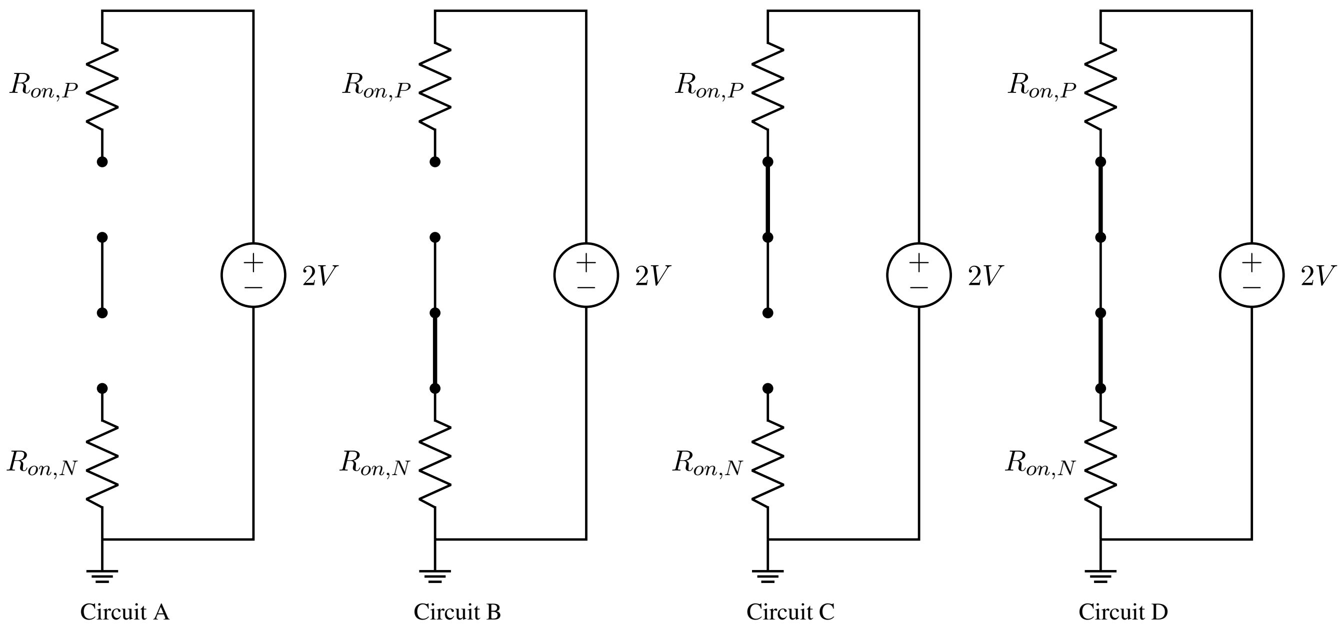
Circuit C

	A	**B**	**C**
Equivalent Circuit	◯	◯	◯

Solution: For the PMOS transistor, $|V_{GS}| = 1.6V > |V_{tp}| = 0.6V$, so the PMOS transistor is on. Thus circuit C is equivalent.

(c) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**





	A	B	C	D
Equivalent Circuit	◯	◯	◯	◯

Solution: For the PMOS transistor, $|V_{GS}| = 1.3V > |V_{tp}| = 0.6V$, so the PMOS transistor is on. For the NMOS transistor, $V_{GS} = 0.7V > V_{tn} = 0.5V$, so the NMOS transistor is on.

Note that in this case, both transistors are on.

Thus circuit D is equivalent.

Aside: In digital logic, it is usually undesirable to have this state in your system for several reasons. First, the output voltage of the inverter (the voltage at the shared drain of the NMOS and PMOS) will not be either 0 or VDD, which means the output voltage is not at 'true' binary value. In addition, we now have a direct current path through the NMOS and PMOS transistors from VDD to ground. This will burn a lot of power! In reality, all inverters briefly transition through this state where both NMOS and PMOS are on when the inputs change from 1 to 0 or 0 to 1.

Contributors:

- Anant Sahai.
- Sidney Buchbinder.
- Pavan Bhargava.