
EECS 16B Designing Information Devices and Systems II
Fall 2021 UC Berkeley Homework 0

This homework is due on Friday, August 27, 2021, at 11:59PM. Self-grades and HW Resubmission are due on Tuesday, August 31, 2021, at 11:59PM.

Problems 4-11 are drawn from the EECS16A Fall 2020 Final Exam. The Final Exam is designed to fit within a 3 hour period. Given the additional problem as well as the logistical questions, the homework should take about 4-5 hours, not accounting for review time.

1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: [Note 0A](#) [Note 0B](#)

- (a) Looking at the different conceptual stages/blocks for dealing with the Cyborg problem, which parts do you think that you are most interested in?
- (b) How do you think that analogous ideas are important in the development of modern high-performance software systems?

2. Surveys

Below we have attached two google form links for you to fill out. The first is a demographic survey that will help us understand more about you and your background. Getting to know you will only help us improve your experience in EECS16B. The second is to indicate to if you would like us to match you with a study group. Participating in study groups is completely optional, please choose an option that is best for you.

- (a) [Demographic Survey - Google Form](#)
- (b) [Group Formation Survey - Google Form](#)

3. Course Policies

[Go to the course website and read the course policies carefully.](#) Leave a followup in the Homework 0, Question 3 thread on Piazza if you have any questions. Are the following situations violations of course policy? Write "Yes" or "No", and a short explanation for each.

- (a) Alice and Bob work on a problem in a study group. They write up a solution together and submit it, noting on their submissions that they wrote up their homework answers together.
- (b) Carol goes to a homework party and listens to Dan describe his approach to a problem on the board, taking notes in the process. She writes up her homework submission from her notes, crediting Dan.
- (c) Erin gets frustrated by the fact that a homework problem given seems to have nothing in the lecture, notes, or discussion that is parallel to it. So, she starts searching for the problem online. She finds a solution to the homework problem on a website. She reads it and then, after she has understood it, writes her own solution using the same approach. She submits the homework with a citation to the website.

- (d) Frank is having trouble with his homework and asks Grace for help. Grace lets Frank look at her written solution. Frank copies it onto his notebook and uses the copy to write and submit his homework, crediting Grace.
- (e) Heidi has completed her homework. Her friend Irene has been working on a homework problem for hours, and asks Heidi for help. Heidi sends Irene her photos of her solution through an instant messaging service, and Irene uses it to write her own solution with a citation to Heidi.

4. Save Baby Yoda!

Despite our best efforts, we have lost Baby Yoda to former agents of the Galactic Empire. Luckily we were able to conceal a receiver in his locket, so now it's time to save Baby Yoda using our 16A knowledge!

- (a) Baby Yoda has been delivered to an Imperial Star Destroyer. Rebel intel has provided us with access to their internal communication beacons. The ship's layout is 2-dimensional with 3 beacon locations specified in Table 1.

Beacon	Coordinates	Distance to Baby Yoda
A	(5,5)	$\sqrt{20}$
B	(2,3)	1
C	(1,1)	2

Table 1: Data from Destroyer Beacons and their coordinates.

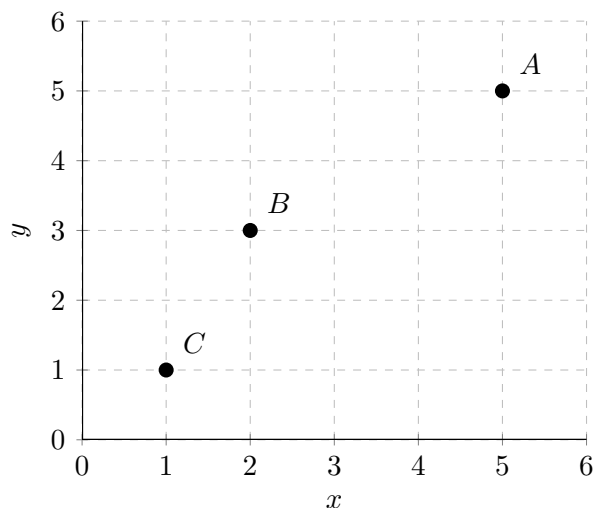


Figure 1: Diagram of the Destroyer's floor-plan with Beacon coordinates marked accordingly.

Explicitly write out a *linear* system of equations (in matrix-vector form) using the data above for finding Baby Yoda's location $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Draw a box around your final linear system, then solve for Baby Yoda's location. Nonlinear terms are not permitted in your final system of equations. You must provide both the system and the location for full credit.

5. Ultrasound Sensing with Op-Amps

The transresistance amplifier is often used to convert a current from a sensor to a voltage. In this problem we will use it to build an ultrasound sensor! When an ultrasonic wave hits our sensor, it generates a current, i_{ultra} . Whenever no ultrasonic wave hits our sensor zero current is generated, so $i_{\text{ultra}} = 0$.

*Note: An **ideal op-amp** is used in all subparts of this question. You can also assume that $V_{DD} = -V_{SS}$.*

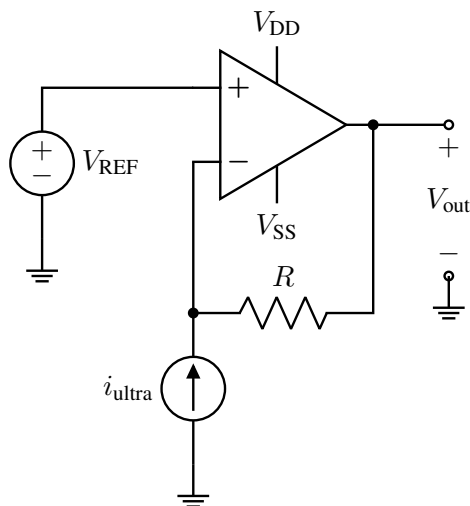


Figure 2: Transresistance sensor circuit

- Calculate the output voltage, V_{out} , of the transresistance sensor circuit shown in Fig. 2, as a function of the reference voltage, V_{REF} , the sensor input current, i_{ultra} , and the resistor, R , when an ultrasonic wave hits the sensor. Clearly show all your work and justify your answer. Writing only the final expression will not be given full credit.
- Assume that the amplitude of the ultrasonic wave hitting the sensor is such that the current i_{ultra} fluctuates from a minimum value of $i_{\text{min}} = 1 \cdot 10^{-6}\text{A}$, to a maximum value of $i_{\text{max}} = 2 \cdot 10^{-6}\text{A}$. Also assume that the reference voltage is set to $V_{\text{REF}} = 1\text{V}$. In this case, calculate the following:
 - The maximum value of the resistor, R , so that the output voltage, V_{out} , does not drop below **0V**. Clearly show all your work.
 - Assuming you picked $R = 250 \cdot 10^3\Omega$ (which may or may not be the correct answer to part (i)), calculate the maximum value of the output voltage, V_{out} . Clearly show all your work.
- Unfortunately, after a few hours of successful ultrasound sensing, our sensor got damaged. It now constantly generates a huge background current, I_{damage} . So when an ultrasonic wave hits it, the sensor produces $I_{\text{damage}} + i_{\text{ultra}}$, as shown in Fig 3(b). When no ultrasonic wave hits it, the sensor produces just I_{damage} . However, the huge background current causes our output to constantly be $V_{\text{out}} = V_{\text{SS}}$, so we are not able to tell whether an ultrasonic wave is present or not.

We would like to fix this in our circuit by canceling the background current and retaining only the useful signal. For this purpose we are going to use a current source, I_{fix} , shown in Fig. 3(a), whose value we can choose. **How would you connect this current source in your circuit and what value would you pick for it?** Redraw the entire circuit with the new current source, I_{fix} , added and give the value of I_{fix} in terms of I_{damage} , i_{ultra} , R , V_{REF} . **Explain how your design works.**

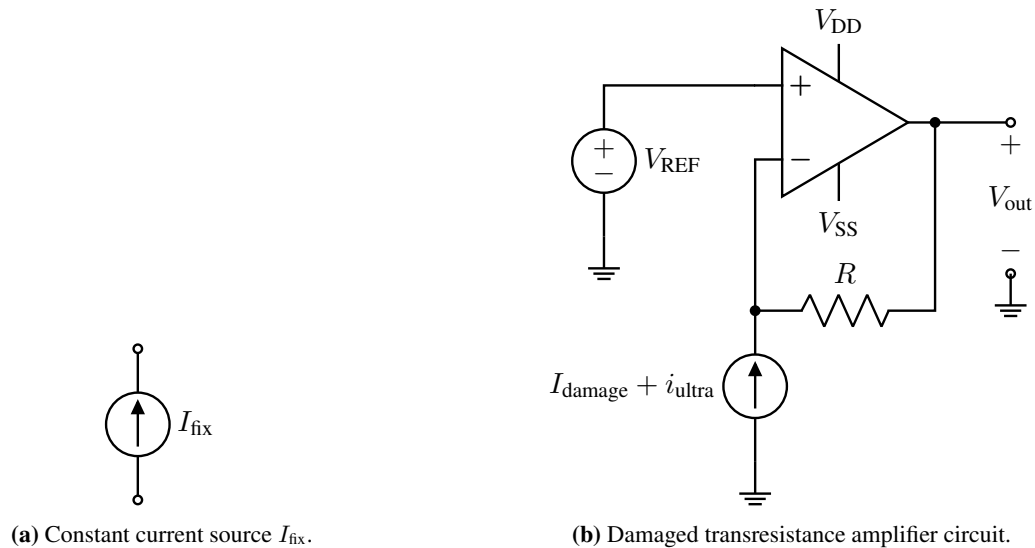


Figure 3: Circuits detailing the transresistance amplifier design, including the background signal I_{damage} .

6. Saving Lives with Op-Amps

An electrocardiogram, or ECG, is a medical device used to detect electrical signals in your heart. Typically, the voltage signal from the human heart is only 1×10^{-3} V at maximum. However, in order for healthcare professionals to properly interpret ECGs, these signals need to be amplified so that abnormalities are more obvious. In this problem we will do so by using **ideal op-amps**.

Note: Assume that $V_{DD} = -V_{SS}$ in all subparts.

- (a) We need to amplify the voltage signal recorded by the electrodes V_{in} by a factor of 1000. Using the op-amp in Figure 4 below and 2 resistors, **draw a circuit that achieves $V_{\text{out}} = 1000 \cdot V_{\text{in}}$. Write an equation for V_{out} in terms of V_{in} and the resistor(s), label the resistor(s) you use (i.e. R_1, R_2), and choose their values.** You should redraw the entire circuit in your answer sheet, but there is no need to draw the human as long as you label V_{in} . Clearly explain and show your work.

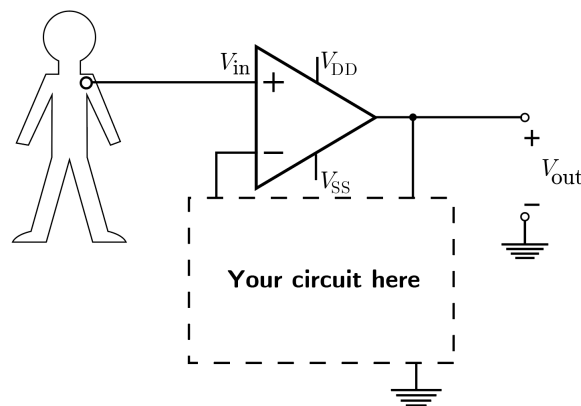


Figure 4: Unfinished ECG amplification circuit.

(b) A friend of yours is also working on an ECG amplification circuit, and shows you their design in Figure 5. Their design uses $R_{\text{electrode}} = 1 \cdot 10^3 \Omega$, $R_1 = 1 \cdot 10^3 \Omega$, and $R_2 = 1 \cdot 10^6 \Omega$. They claim their circuit gives, $V_{\text{out}} = -1000 \cdot V_{\text{in}}$. **Is their claim true?**

- **If yes, justify why.**
- **If no, how would you choose the value of R_2 to achieve $V_{\text{out}} = -1000 \cdot V_{\text{in}}$, assuming that both $R_{\text{electrode}}$ and R_1 are fixed at $R_{\text{electrode}} = R_1 = 1 \times 10^3 \Omega$? Clearly show your work, and justify your answers.**

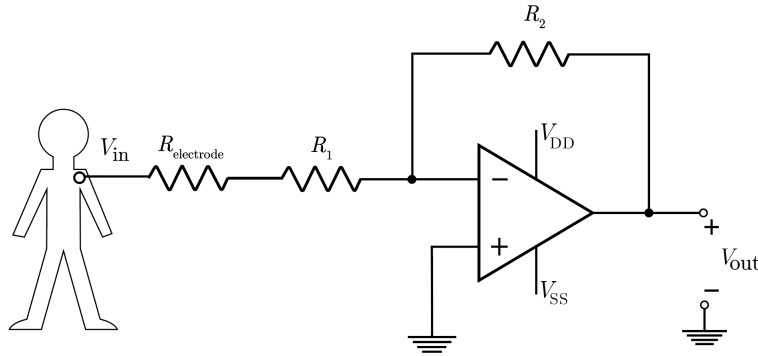


Figure 5: An alternative ECG op-amp circuit.

(c) Another configuration often used by healthcare professionals is to attach one electrode to the heart (recording its electrical signal, V_{in}) and another electrode to the right leg to serve as a reference voltage, as shown in Figure 6. **What is the output voltage, V_{out} , as a function of V_{in} , V_{RL} , R_{bottom} , and R_{top} ? Clearly show your work.**

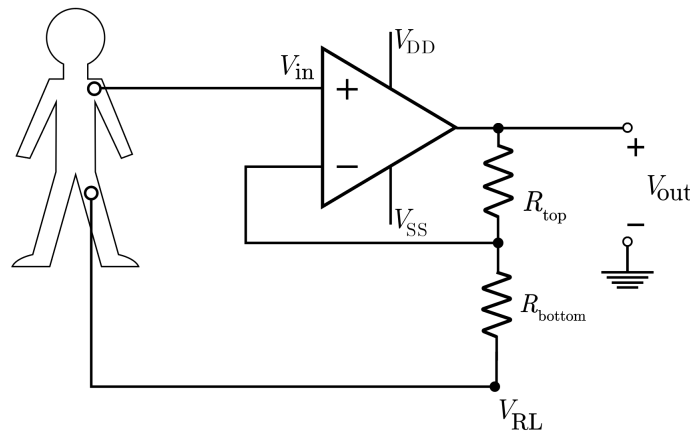


Figure 6: Alternative op-amp ECG topology.

- (d) Even after amplification, certain peaks of your ECG signal are too low to be discerned. You want to sample them and amplify them a bit more. To this end, you use the circuit in Figure 7. The circuit cycles through two phases: in phase 1, switches labeled ϕ_1 are ON and ϕ_2 are OFF, while in phase 2, switches labeled ϕ_2 are ON and ϕ_1 are OFF. Calculate the output voltage, V_{out} , during phase 2, after steady state has been reached, in terms of C_1 , C_2 and V_{in} . Clearly show your work.

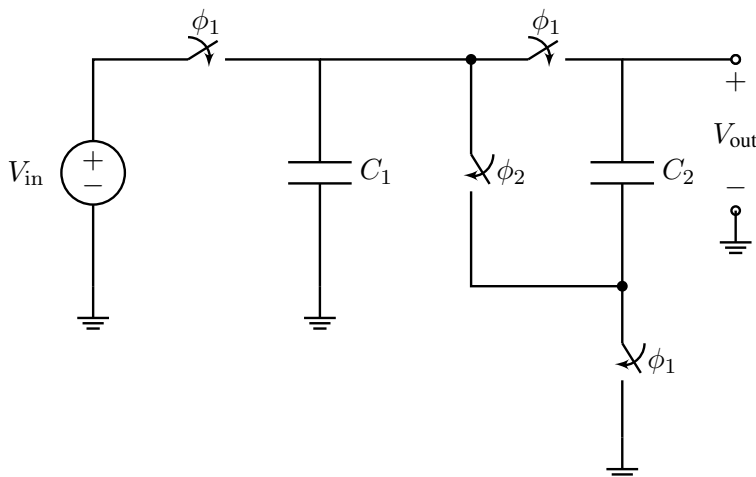


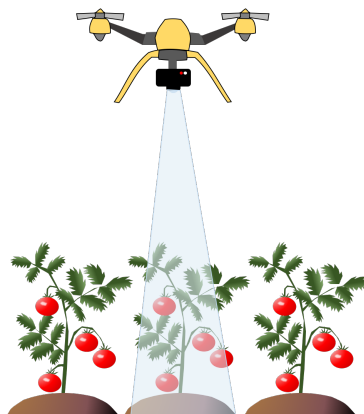
Figure 7: Switch capacitor voltage boosting circuit.

7. Hyperspectral Classification of Tomatoes

You're a high-tech farmer who just bought a new hyperspectral sensor to monitor your crops.

NOTE: You do not need to understand how a hyperspectral sensor works to solve this problem.

You attach the sensor to a drone and fly it over your crops, taking measurements of the hyperspectral signature for different points along the field. You want to use these measurements to identify which crops are healthy and which crops are getting sick. Your sensor gives you a spectral signature for each plant as a length 5 vector, where each entry of the vector represents a different frequency. Scientists have determined that healthy versus sick tomato plants will have the following spectral signatures as shown in Figure 8.



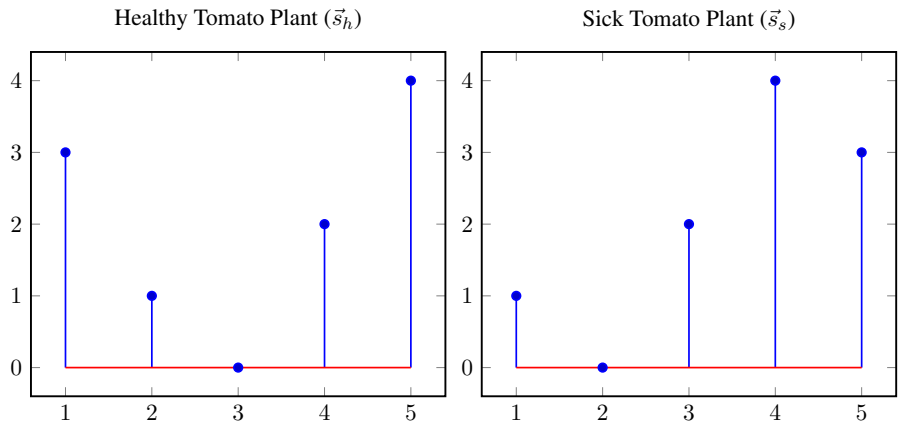


Figure 8: Spectral signature for healthy tomato plant (\vec{s}_h) and sick tomato plant (\vec{s}_s).

They can also be represented in vector form:

$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

- (a) Using your spectral sensor, you measure the following spectral signature for one of your tomato plants as shown in Figure 9. This measurement has some noise in it.

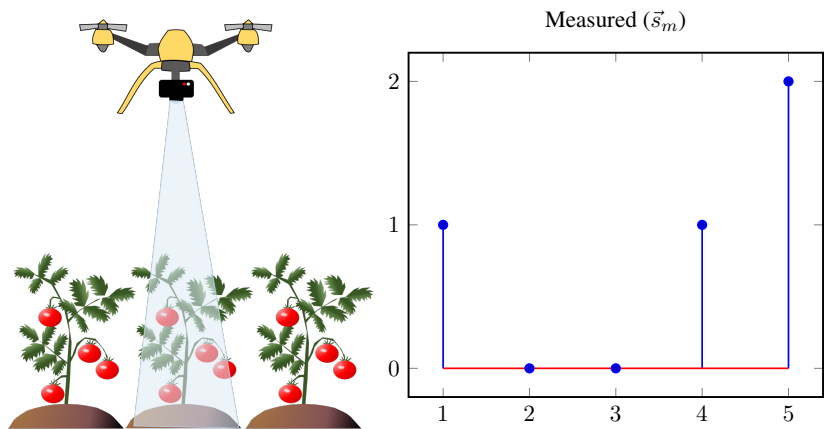


Figure 9: Spectral signature for the measurement, \vec{s}_m .

The spectral signatures for healthy, sick, and your measured tomato plants can also be represented in vector form as

$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{s}_m = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Since spectral signatures never exactly match, the standard procedure is to calculate the angle between signature vectors to determine how close they are. **Compute the angle between \vec{s}_m and \vec{s}_h and the angle between \vec{s}_m and \vec{s}_s . Is your measured vector closer to the sick plants or the healthy plants?**

Classify your plant's health based on the angle between your measured spectral signature (\vec{s}_m) and the known spectral signatures, (\vec{s}_h, \vec{s}_s). Show your work and justify your answer.

NOTE: Table 2 can be helpful for finding the angles.

Table 2: Cosine Table

$\cos(\theta)$	$\theta(^{\circ})$
$\frac{9}{\sqrt{180}}$	47.87
$\frac{10}{\sqrt{180}}$	41.81
$\frac{11}{\sqrt{180}}$	34.93
$\frac{12}{\sqrt{180}}$	26.57
$\frac{13}{\sqrt{180}}$	14.31

- (b) It's a windy day and the drone got pushed as it was taking a measurement, so now the measurement has a linear combination of measurements for several different tomato plants (some of which are healthy and some of which are sick). So your measurement is

$$\vec{s}_m = \alpha\vec{s}_h + \beta\vec{s}_s + \vec{e}, \quad (1)$$

where \vec{e} represents an error vector that is unknown.

The values you get for your measurement are:

$$\vec{s}_m = \begin{bmatrix} 5 \\ 1 \\ 4 \\ 10 \\ 10 \end{bmatrix}.$$

The measurement is also shown in Figure 10.

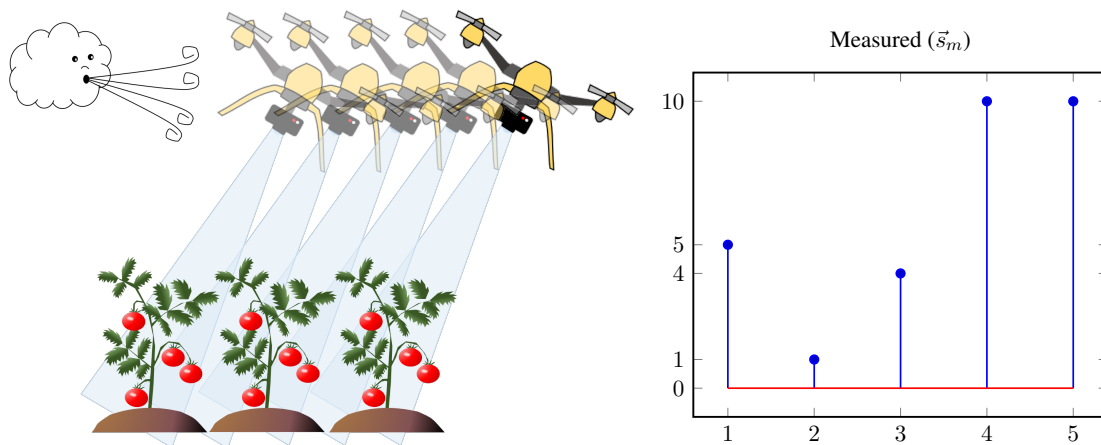


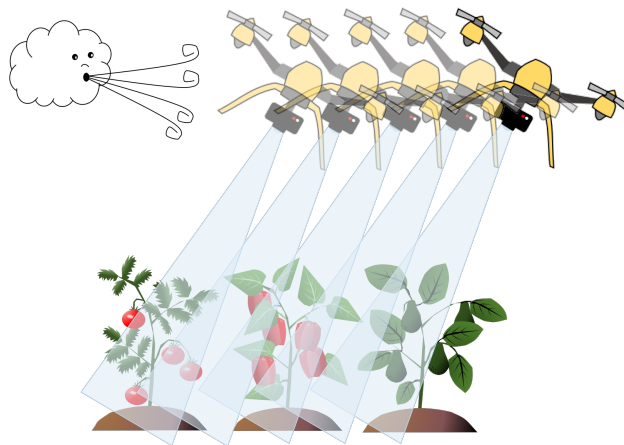
Figure 10: Spectral signature for your measurement, \vec{s}_m .

Recall that

$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

You want to identify the unknowns α and β . **Write a least squares problem in the format $A\vec{x} = \vec{b}$ to identify the unknowns α and β . Show your work.** You do not have to solve for α and β .

- (c) Your drone got pushed by the wind again, but this time it was while it was taking a measurement on the border of three adjacent fields - your tomato, pepper, and avocado fields.



Tomato, pepper, and avocado plants have unique spectral signatures with a length of 5. The notations are described as the following:

- \vec{s}_h and \vec{s}_s represent the spectral signatures of healthy and sick tomato plants
- \vec{s}_{ph} and \vec{s}_{ps} represent the spectral signatures of healthy and sick pepper plants
- \vec{s}_{ah} and \vec{s}_{as} represent the spectral signatures of healthy and sick avocado plants

Your measurement is now a linear combination of 6 possible spectral signatures:

$$\vec{s}_m = \alpha_1 \vec{s}_h + \beta_1 \vec{s}_s + \alpha_2 \vec{s}_{ph} + \beta_2 \vec{s}_{ps} + \alpha_3 \vec{s}_{ah} + \beta_3 \vec{s}_{as}. \quad (2)$$

Here $\alpha_1, \alpha_2, \alpha_3$ are the unknown weights of healthy tomato, pepper, and avocado plants respectively. $\beta_1, \beta_2, \beta_3$ are the unknown weights of sick tomato, pepper, and avocado plants respectively. **Is it possible to uniquely determine the weights of healthy/sick tomatoes, peppers, and avocados from your measurement in equation 2? Why or why not? Show your work and justify your answer.**

8. Cross-correlation

We are building our own Acoustic Positioning System.

NOTE: The signatures \vec{s}_1, \vec{s}_2 in each sub-part are different; each prompt is independent from the others.

- (a) We have two signatures/gold codes of length-5, given by \vec{s}_1 and \vec{s}_2 as in Figure 11. So far we have numerically computed their linear cross-correlation $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$, yet a few entries have been tragically lost! Fortunately we can compute these omitted terms by hand. **Please compute the missing cross-correlation values at shifts $k = -1$ and $k = +2$. Show your work and justify your answer.**

$$\vec{s}_1 = \begin{bmatrix} +1 \\ 0 \\ -1 \\ 0 \\ +1 \end{bmatrix} \quad \vec{s}_2 = \begin{bmatrix} +1 \\ +1 \\ 0 \\ -1 \\ +1 \end{bmatrix}.$$

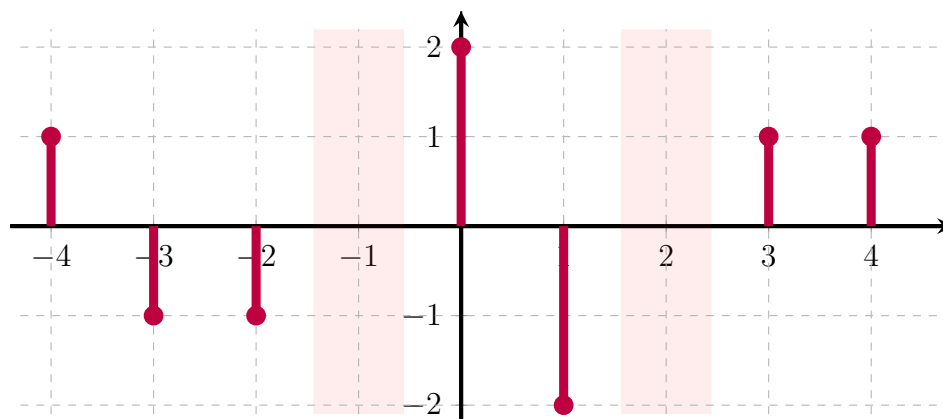


Figure 11: Linear cross-correlation plot of the two signals $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$. The x-axis represents the shift.

- (b) We are trying out some new codes \vec{s}_1 and \vec{s}_2 . We only know that the codes are normalized ($\langle \vec{s}_1, \vec{s}_1 \rangle = 1$, $\langle \vec{s}_2, \vec{s}_2 \rangle = 1$) and their inner-product is $\langle \vec{s}_1, \vec{s}_2 \rangle = 0.3$. During our test we have received the signal $\vec{r} = \frac{1}{2}\vec{s}_1 + \frac{1}{3}\vec{s}_2$. Without knowing any more information about our codes, **compute $\text{Corr}_{\vec{r}}(\vec{s}_1)$ at the shift $k = 0$. Show your work and justify your answer.**
- (c) We again have two new signals \vec{s}_1 and \vec{s}_2 and are now given the plot of $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$ as shown in Figure 12. Our receiver identified a signal \vec{r} which we know to be related to the code \vec{s}_2 by some scaling, shifting, and/or reflection. However, we only know the linear cross-correlation $\text{Corr}_{\vec{s}_1}(\vec{r})$ as shown in Figure 13. **Can you express \vec{r} in terms of \vec{s}_2 ? Show your work and justify your answer.**

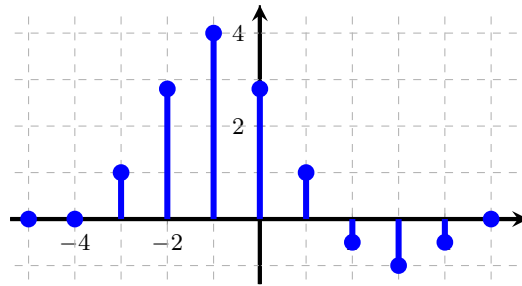


Figure 12: Linear cross-correlation plots for $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$.

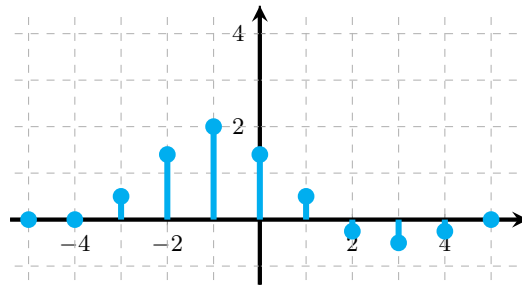


Figure 13: Linear cross-correlation plots for $\text{Corr}_{\vec{s}_1}(\vec{r})$.

- (d) With a little effort, we managed to create two *good* gold codes of length 100, \vec{s}_1 and \vec{s}_2 . The linear cross-correlation of \vec{s}_1 and \vec{s}_2 is small at all shifts while the autocorrelation of each signal is also small, except at shift $k = 0$. We receive our first signal \vec{r} , which we know to be a combination of both codes, of the following form:

$$\vec{r}[n] = \vec{s}_1[n - k_1] + \vec{s}_2[n - k_2]. \tag{3}$$

The linear cross-correlation $\text{Corr}_{\vec{r}}(\vec{s}_1)$ has been computed and plotted in Figure 14, and similarly $\text{Corr}_{\vec{r}}(\vec{s}_2)$ is plotted in Figure 15. **Determine the shifts for \vec{s}_1 and \vec{s}_2 in the received signal \vec{r} , i.e. solve for k_1 and k_2 in equation (3). Explain your answer.**

Note: Don't worry too much about identifying the exact value for k_1 and k_2 . As long as your answer is reasonable, you will receive full credit.

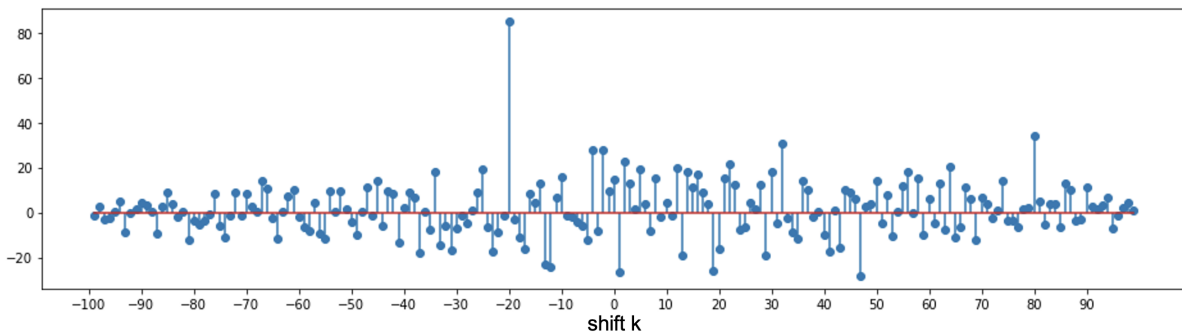


Figure 14: Linear cross-correlation plots for $\text{Corr}_{\vec{r}}(\vec{s}_1)$.

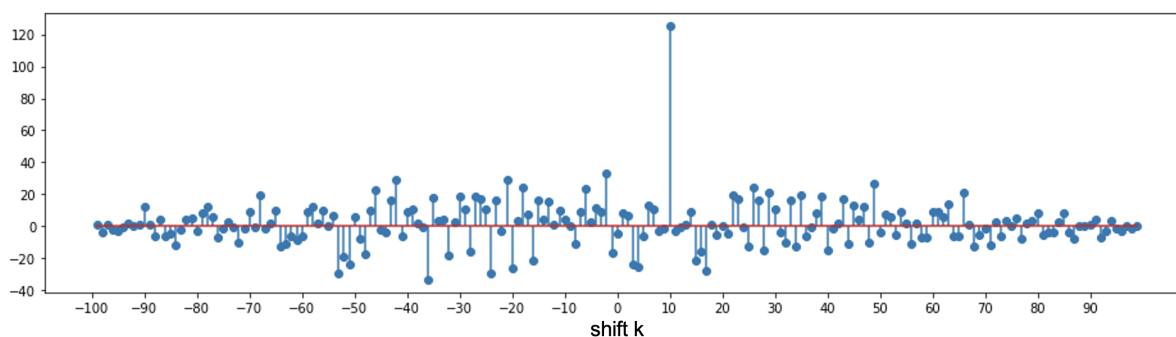


Figure 15: Linear cross-correlation plots for $\text{Corr}_T(\vec{s}_2)$.

- (e) It appears that making codes orthogonal to each other improves the robustness of our Acoustic Positioning System. Knowing this, we want to use our knowledge of projections to write our first code as $\vec{s}_1 = \vec{a} + \vec{b}$, where $\langle \vec{b}, \vec{s}_2 \rangle = 0$ and $\vec{a} = \alpha \vec{s}_2$ (for some constant α) as illustrated in Figure 16. **Compute α and \vec{b} in terms of \vec{s}_1 and \vec{s}_2 . Show your work and justify your answer.**

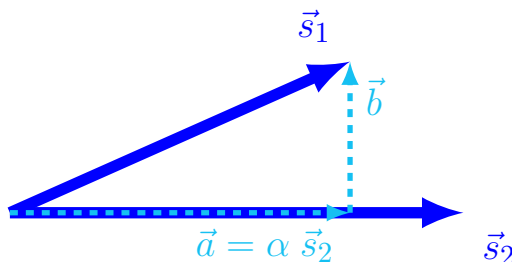


Figure 16: 2D figure of $\vec{s}_1 = \vec{a} + \vec{b}$.

- (f) After optimizing two orthogonal codes \vec{s}_1 and \vec{s}_2 (i.e. $\langle \vec{s}_1, \vec{s}_2 \rangle = 0$), we would next like to include another code \vec{s}_3 and make it orthogonal to \vec{s}_1 and \vec{s}_2 . We can start by writing \vec{s}_3 as $\vec{s}_3 = \vec{a} + \vec{b}$, such that \vec{a} belongs to the span $\{\vec{s}_1, \vec{s}_2\}$ and \vec{b} is orthogonal to span $\{\vec{s}_1, \vec{s}_2\}$, i.e. $\langle \vec{b}, \vec{s}_1 \rangle = 0$ and $\langle \vec{b}, \vec{s}_2 \rangle = 0$. **Use the idea of projections to write both \vec{a} and \vec{b} in terms of \vec{s}_1 , \vec{s}_2 , and \vec{s}_3 , and inner-products thereof. (For full credit your final answer may not contain matrices nor matrix-vector products).** Show your work and justify your answer.

9. Warm for the Holidays

Winter is coming, and both you and your roommate are in desperate need of electric heating eye pads to avoid overly dry eyes this holiday. Tragically the circuit for your eye pads broke, yet fortunately you've taken EECS16A and have come up with a clever fix by designing a voltage divider and a comparator circuit!

- (a) First you build a circuit that converts temperature change to voltage change. Your design is shown in Fig. 17. In your design you use two temperature dependent resistors, whose values are given by $R_0 + \alpha T$, $R_0 - \alpha T$, where R_0 is the resistor value at 0 degrees centigrade, α is a thermal coefficient, and T is the temperature of your eye pads.
What is the temperature dependent output voltage, V_T , of this circuit, as a function of V_s , R_0 , α , and T ? Is V_T a linear function of T ? Clearly show all your work.
- (b) We want to use a comparator to turn the heat ON and OFF, and you set up the circuit in Fig. 18. You process the V_T to make $V_{in} = (1 - \frac{T}{T_0})[\text{Volts}]$, where $T_0 = 30^\circ\text{C}$. The heat will turn on when

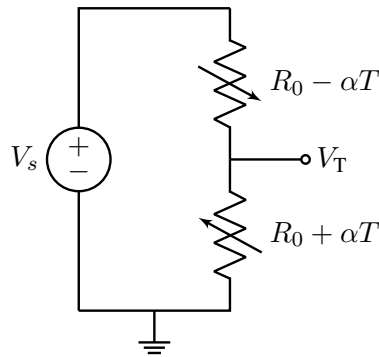


Figure 17: Temperature Sensing Circuit

$V_{out} = V_{DD}$. For what range of temperatures, T , is $V_{out} = V_{DD}$? Give your answer in terms of $^{\circ}\text{C}$. Clearly show all your work.

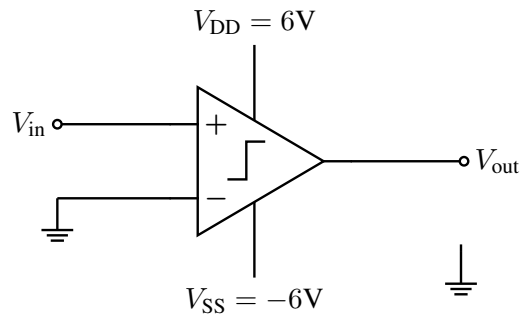


Figure 18: First attempt eye-pad control circuit.

- (c) Your TA, Moses, points out that just using the circuit in Figure 18 will cause your heat to turn ON and OFF due to very small fluctuations. Instead, he suggests analyzing the following circuit in Figure 19. Find the voltage u_+ at the positive terminal of the comparator, as a function of V_{out} , R_1 , R_2 , and V_{ref} . Clearly show all your work.

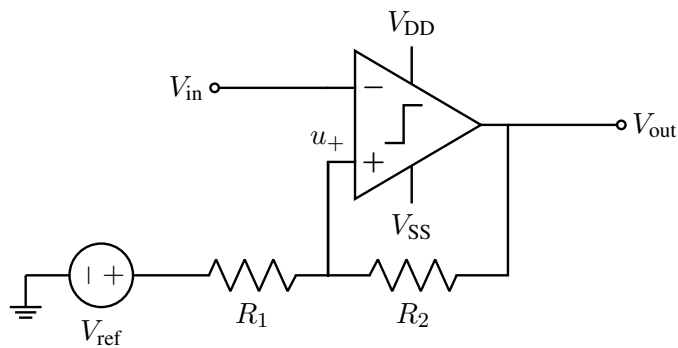


Figure 19: Proposed eye-pad control circuit.

10. Least Squares for Robotics

Robots rely on sensors for understanding their environment and navigating in the real world. These sensors must be calibrated to ensure accurate measurements, which we explore in this problem.

- (a) Your robot is equipped with two forward-facing sensors – a radar and camera.

However, the sensors are placed with an offset (i.e. a gap) of ℓ in meters (m), as depicted in Fig. 20, and you want to find its value. The radar returns a range ρ in meters (m) and heading angle θ in radians (rad) with respect to the object. In contrast, the camera only returns an angle, ϕ in radians (rad), with respect to the object.

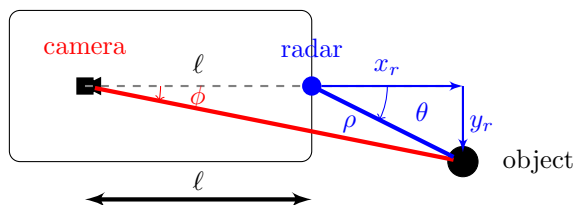


Figure 20: Sensor Placement and Offset ℓ .

These relationships are summarized by the following sensor model, where x_r and y_r are the Cartesian coordinates of the object with respect to the radar:

$$x_r = \rho \cos(\theta), \quad (4)$$

$$y_r = \rho \sin(\theta), \quad (5)$$

$$\tan(\phi) = \frac{y_r}{x_r + \ell}. \quad (6)$$

Assuming $\phi \neq 0$, use equations (4), (5), (6) to express ℓ in terms of ρ , θ , and ϕ .

- (b) Often it is difficult to precisely identify the value of ℓ . To learn the value of ℓ you decide to take a series of measurements. In particular, you take N measurements and get the equations:

$$a \ell + e_i = b_i$$

for $1 \leq i \leq N$. Here $a \neq 0$ is a fixed and known constant. Each b_i represents your i^{th} measurement and e_i represents the error in your measurement. While you know all of the b_i values, you do not know the error values e_i .

We can write this equation in a vector format as:

$$\mathbf{A} \ell + \vec{e} = \vec{b},$$

$$\text{where } \mathbf{A} = \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}, \vec{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}.$$

In this simple 1-D case, the least squares solution is a scaled version of the average of $\{b_i\}_{i=1}^N$.

Find the best estimate for ℓ , denoted as $\hat{\ell}$, using least squares. Simplify your expression and express $\hat{\ell}$ in terms of a , b_i , and N . Your answer may not include any vector notation.

Note: A is a vector and not a matrix.

(c) Now we turn to the task of controlling the robot's velocity and acceleration, which is a key requirement for navigation.

We use the following model for the robot, which describes how the velocity and acceleration of the robot changes with timestep k :

$$\begin{bmatrix} v[k+1] \\ a[k+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v[k] \\ a[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} j[k],$$

where

- k is the timestep;
- $v[k]$ is the velocity state at timestep k ;
- $a[k]$ is the acceleration state at timestep k ;
- $j[k]$ is the jerk (derivative of acceleration) control input at timestep k .

We start at a known initial state $\begin{bmatrix} v[0] \\ a[0] \end{bmatrix}$, and we want to find $j[0]$ to set $\begin{bmatrix} v[1] \\ a[1] \end{bmatrix}$ as close to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as possible. For this, we minimize:

$$E = \left\| \begin{bmatrix} v[1] \\ a[1] \end{bmatrix} \right\|^2.$$

Find the best estimate for the optimal choice of jerk, $\hat{j}[0]$, by using least squares method to minimize E . Express your solution in terms of $v[0]$ and $a[0]$. Show your work.

Hint: Rewrite E in terms of $j[0]$ and other relevant terms.

11. Proof

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. The eigenvalues and eigenvectors of \mathbf{A} are given by $(\alpha_1, \vec{v}_1), (\alpha_2, \vec{v}_2), \dots, (\alpha_n, \vec{v}_n)$, where all the α_i , $1 \leq i \leq n$, are distinct. Similarly the eigenvalues and eigenvectors of \mathbf{B} are given by $(\beta_1, \vec{v}_1), (\beta_2, \vec{v}_2), \dots, (\beta_n, \vec{v}_n)$, where all the β_i , $1 \leq i \leq n$, are distinct.

NOTE: \mathbf{A}, \mathbf{B} have identical eigenvectors.

Prove that:

$$\mathbf{A} \mathbf{B} \vec{x} = \mathbf{B} \mathbf{A} \vec{x},$$

for any vector $\vec{x} \in \mathbb{R}^n$.

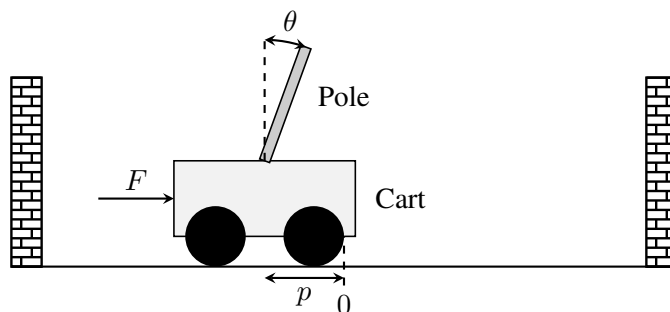
12. Segway Tours

Learning Objective: *The learning objective of this problem is to see how the concept of span can be applied to control problems. If a desired state vector of a linear control problem is in a span of a particular set of vectors, then the system may be steered to reach that particular vector using the available inputs.*

Your friends have decided to start a new SF tour business, and you suggest they use segways. They become intrigued by your idea and asks you how a segway works. A segway is essentially a stand on two wheels.

The segway works by applying a force (through the spinning wheels) to the base of the segway. This controls both the position on the segway and the angle of the stand. As the driver pushes on the stand, the segway tries to bring itself back to the upright position, and it can only do this by moving the base.

Is it possible for the segway to be brought upright and to a stop from any initial configuration? There is only one input (force) used to control two outputs (position and angle). You talk to a friend who is GSling EE128, and she tells you that a segway can be modeled as a cart-pole system.



A cart-pole system can be fully described by its position p , velocity \dot{p} , angle θ , and angular velocity $\dot{\theta}$. We write this as a “state vector”, \vec{x} :

$$\vec{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}.$$

The input to this system is a scalar quantity $u[n]$ at time n , which is the force F applied to the cart (or base of the segway).¹

The cart-pole system can be represented by a linear model:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n], \quad (7)$$

where $\mathbf{A} \in \mathbb{R}^{4 \times 4}$ and $\vec{b} \in \mathbb{R}^{4 \times 1}$.

The control $u[n]$ allows us to move the state (\vec{x}) in the direction of \vec{b} . So, if $u[n] = 2$, we move the state by $2\vec{b}$ at time n , and so on. We can choose different controls at different times.

The model tells us how the state vector, \vec{x} , will evolve over time as a function of the current state vector and control inputs.

You look at this general linear system and try to answer the following question: Starting from some initial state \vec{x}_0 , can we reach a final desired state, \vec{x}_f , in N steps?

The challenge seems to be that the state is four-dimensional and keeps evolving and that we can only apply a one-dimensional (scalar) control at each time. Typically, to set the values of four variables to desired quantities, you would need four inputs. Can you do this with just one input?

We will solve this problem by walking through several steps.

- (a) Express $\vec{x}[1]$ in terms of $\vec{x}[0]$ and the input $u[0]$.

¹You might note that velocity and angular velocity are derivatives of position and angle respectively. Differential equations are used to describe continuous time systems, which you will learn more about in EECS 16B. But even without these techniques, we can still approximate the solution to be a continuous time system by modeling it as a discrete time system where we take very small steps in time. We think about applying a force constantly for a given finite duration and we see how the system responds after that finite duration.

- (b) i. Express $\vec{x}[2]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$ and $u[1]$.
 ii. Then express $\vec{x}[3]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, and $u[2]$.
 iii. Finally express $\vec{x}[4]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, $u[2]$, and $u[3]$.
Your expressions can have other relevant variables (e.g. \mathbf{A} , \vec{b} etc) and mathematical operators.
- (c) Now, generalize the pattern you saw in the earlier part to write an expression for $\vec{x}[N]$ in terms of $\vec{x}[0]$ and the inputs from $u[0], \dots, u[N-1]$. **Your expression can have other relevant variables (e.g. \mathbf{A} , \vec{b} etc) and mathematical operators.**

For the next four parts of the problem, you are given the matrix \mathbf{A} and the vector \vec{b} :

$$\mathbf{A} = \begin{bmatrix} 1 & 0.05 & -0.01 & 0 \\ 0 & 0.22 & -0.17 & -0.01 \\ 0 & 0.10 & 1.14 & 0.10 \\ 0 & 1.66 & 2.85 & 1.14 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0.01 \\ 0.21 \\ -0.03 \\ -0.44 \end{bmatrix}$$

Assume the cart-pole starts in an initial state $\vec{x}[0] = \begin{bmatrix} -0.3853493 \\ 6.1032227 \\ 0.8120005 \\ -14 \end{bmatrix}$, and you want to reach the desired

state $\vec{x}_f = \vec{0}$ using the control inputs $u[0], u[1], \dots$ etc. The state vector $\vec{x}_f = \vec{0}$ corresponds to the cart-pole (or segway) being upright and stopped at the origin. **Reaching $\vec{x}_f = \vec{0}$ in N steps means that, given that we start at $\vec{x}[0]$, we can find control inputs ($u[0], u[1], \dots$ etc), such that we get $\vec{x}[N]$ (i.e. state vector at N th time step) equal to $\vec{x}_f = \vec{0}$.**

Note: Please use the Jupyter notebook to solve parts (d) - (g) of the problem. You may use the function we provided `gauss_elim(matrix)` to help you find the **upper triangular form** of matrices. An example of Gaussian Elimination using (`gauss_elim(matrix)`) is provided in the Jupyter notebook under section **Example Usage of `gauss_elim`**. You may also use the function (`np.linalg.solve`) to solve the equations.

- (d) Can you reach \vec{x}_f in *two* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations.
(Hint: Express $\vec{x}[2] - \mathbf{A}^2\vec{x}[0]$ in terms of the inputs $u[0]$ and $u[1]$. Then determine if the system of equations can be solved to obtain $u[0]$ and $u[1]$. If we obtain valid solutions for $u[0]$ and $u[1]$, then we can say we will reach \vec{x}_f in two time steps. Use the notebook to see if the system of equations can be solved.)
- (e) Can you reach \vec{x}_f in *three* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations.
(Hint: Similar to the last part, express $\vec{x}[3] - \mathbf{A}^3\vec{x}[0]$ in terms of the inputs $u[0]$, $u[1]$ and $u[2]$. Then determine if we can obtain valid solutions for $u[0]$, $u[1]$ and $u[2]$.)
- (f) Can you reach \vec{x}_f in *four* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations. *(Use the hints from the last two parts.)*

- (g) If you have found that you can get to the final state in 4 time steps, find the required correct control inputs, i.e. $u[0]$, $u[1]$, $u[2]$ and $u[3]$, using Jupyter and verify the answer by entering these control inputs into the **Plug in your controller** section of the code in the Jupyter notebook. You need to just show that you reached the desired final state \vec{x}_f by plugging in the control inputs. The code has been already written to simulate this system.

Suggestion: See what happens if you enter all four control inputs equal to 0. This gives you an idea of how the system naturally evolves!

- (h) Let us reflect on what we just did. Recall the system we have:

$$\vec{x}[n + 1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n].$$

The control allows us to move the state at time step $n + 1$ by $u[n]$ in direction \vec{b} , remember $u[n] \in \mathbb{R}$ is just a scalar. We know from part (c) that:

$$\vec{x}[2] = \mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1].$$

Again, here $u[0], u[1] \in \mathbb{R}$ can be thought of as arbitrary scalars, and $\mathbf{A}\vec{b}u[0] + \vec{b}u[1]$ can be thought of as the set of all linear combinations of the vectors \vec{b} and $\mathbf{A}\vec{b}$. Using this observation, can you express the possible states you can arrive at in two time steps using the span of exactly *two* vectors plus a vector offset?

- (i) Let's try to generalize the idea in the previous part. Express the states you can reach in N timesteps as a span of some vectors plus a vector offset. (Hint: Consider the direction that each control input $u[0], \dots, u[N - 1]$ can move $\vec{x}[N]$ by.)
- (j) **(OPTIONAL)** Now say you wanted to reach anywhere in \mathbb{R}^4 , i.e. \vec{x}_f is an unspecified vector in \mathbb{R}^4 . Under what conditions can you guarantee that you can “reach” \vec{x}_f from any \vec{x}_0 in N time steps? Wouldn't this be cool?

13. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

- (a) **What sources (if any) did you use as you worked through the homework?**
- (b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)
- (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**