EECS 16B Fall 2021 — Designing Information Devices and Systems II, UC Berkeley

Homework 1

This homework is due on Friday, September 3, 2021, at 11:59PM. Self-grades and HW Resubmissions are due on Tuesday, September 7, 2021, at 11:59PM.

1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note j, Note 1

(a) Have you seen the vector representation of complex numbers in Note j before? Question 3 explores the topics in more details.

(b) Have you solved differential equations of this form before?

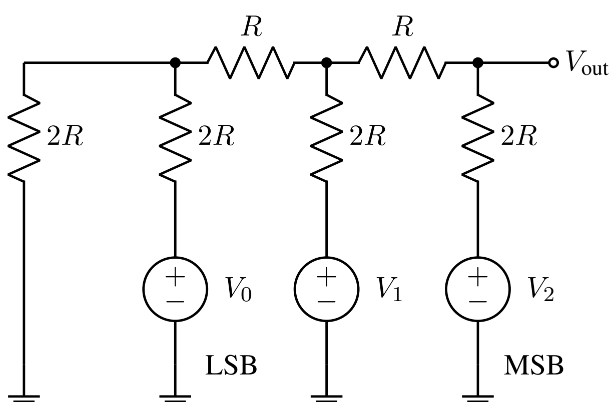
$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{1}$$

Questions 6 and 7 explores the topics of Note 1 in more details.

2. Digital-Analog Converter

A digital-analog converter (DAC) is one of the key interface components between the digital and the analog world. It is a circuit for converting a digital representation of a number (binary) into a corresponding analog voltage. In this problem, we will consider a DAC made out of resistors only (resistive DAC) called the R-$2R$ ladder. This DAC will help us generate the analog voltages from the digital representation, and later will also help us digitize the analog voltages when we will be building analog to digital interfaces in Lab 3, in part based on this ladder-DAC.

Here is the circuit for a 3-bit resistive DAC.



Let $b_0, b_1, b_2 = \{0, 1\}$ (that is, either 1 or 0), and let the voltage sources $V_0 = b_0 V_{\text{DD}}$, $V_1 = b_1 V_{\text{DD}}$, $V_2 = b_2 V_{\text{DD}}$, where V_{DD} is the supply voltage.

As you may have noticed, (b_2, b_1, b_0) represents a 3-bit binary (unsigned) number where each of b_i is a binary bit. b_0 is the least significant bit (LSB) and b_2 is the most significant bit (MSB). We will now analyze how this converter functions.

(a) If $b_2, b_1, b_0 = 1, 0, 0$, what is V_{out}? Express your answer in terms of V_{DD}.

(b) If $b_2, b_1, b_0 = 0, 1, 0$, what is V_{out}? Express your answer in terms of V_{DD}.

(c) If $b_2, b_1, b_0 = 0, 0, 1$, what is V_{out}? Express your answer in terms of V_{DD}.

(d) If $b_2, b_1, b_0 = 1, 1, 1$, what is V_{out}? Express your answer in terms of V_{DD}.

(e) Finally, solve for V_{out} in terms of V_{DD} and the binary bits b_2, b_1, b_0.

(f) Explain how your results above show that the resistive DAC converts the 3-bit binary number (b_2, b_1, b_0) to the output analog voltage V_{out}.

3. Complex Numbers

A common way to visualize complex numbers is to use the complex plane. Recall that a complex number z is often represented in Cartesian form.

$$z = x + \mathrm{j}y \text{ with } \operatorname{Re}\{z\} = x \text{ and } \operatorname{Im}\{z\} = y \tag{2}$$

See Figure 1 for a visualization of z in the complex plane.

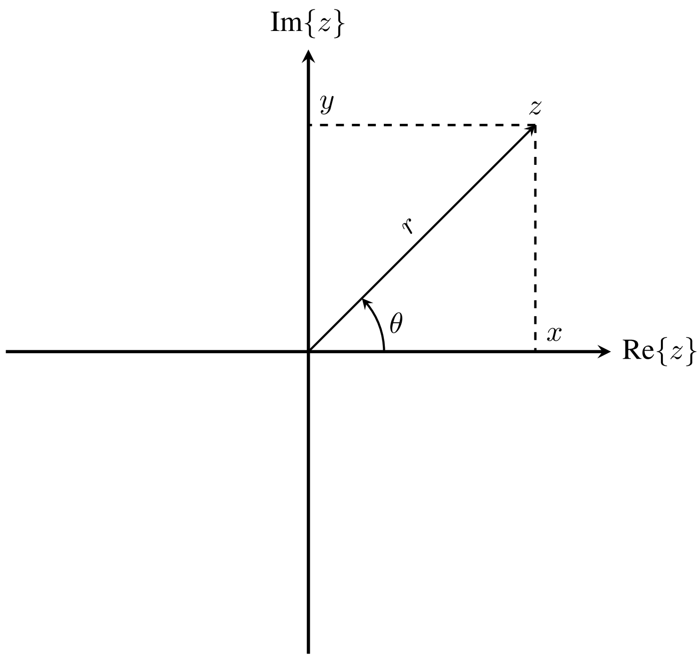


Figure 1: Complex Plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

(a) **Calculate the length of z in terms of x and y as shown in Figure 1.** This is the magnitude of a complex number and is denoted by $|z|$ or r.
(Hint: Use the Pythagorean theorem.)

(b) **Represent x, the real part of z, and y, the imaginary part of z, in terms of r and θ.**

(c) **Substitute for x and y in z.** Use Euler's identity[1] $\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$ to **conclude that,**

$$z = r\mathrm{e}^{\mathrm{j}\theta}. \tag{3}$$

(d) In the complex plane, **sketch the set of all the complex numbers such that $|z| = 1$. What are the z values where the sketched figure intersects the real axis and the imaginary axis?**

(e) If $z = r\mathrm{e}^{\mathrm{j}\theta}$, **prove that** $\overline{z} = r\mathrm{e}^{-\mathrm{j}\theta}$. Recall that the complex conjugate of a complex number $z = x + \mathrm{j}y$ is $\overline{z} = x - \mathrm{j}y$.

[1] also known as de Moivre's Theorem.

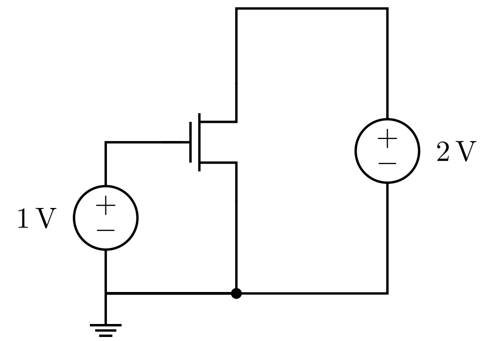
(f) **Show (by direct calculation) that,**

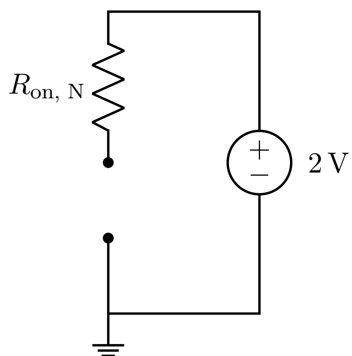
$$r^2 = z\overline{z}. \tag{4}$$

4. Transistor Behavior

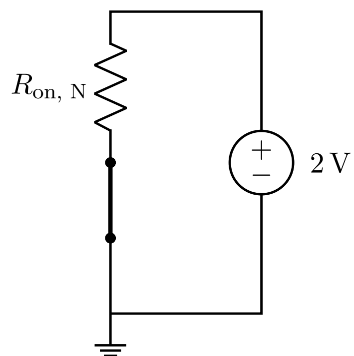
For all NMOS devices in this problem, $V_{\mathrm{tn}} = 0.5\,\mathrm{V}$. For all PMOS devices in this problem, $|V_{\mathrm{tp}}| = 0.6\,\mathrm{V}$.

(a) Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**





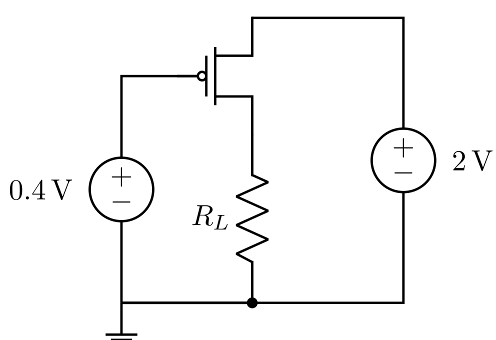
Circuit A

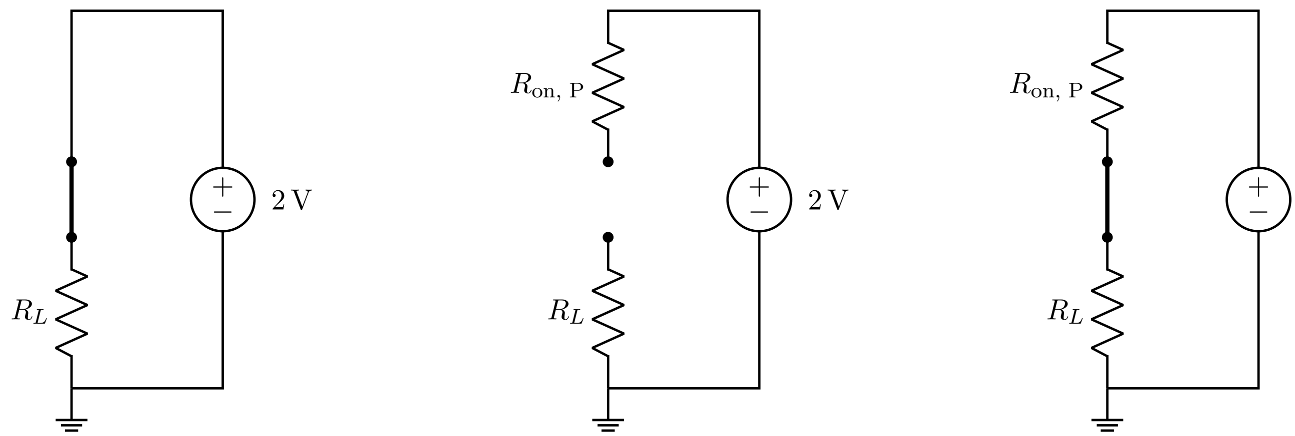


Circuit B

	A	**B**
Equivalent Circuit	◯	◯

(b) Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**

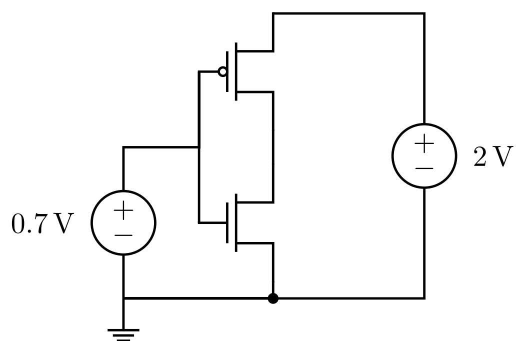


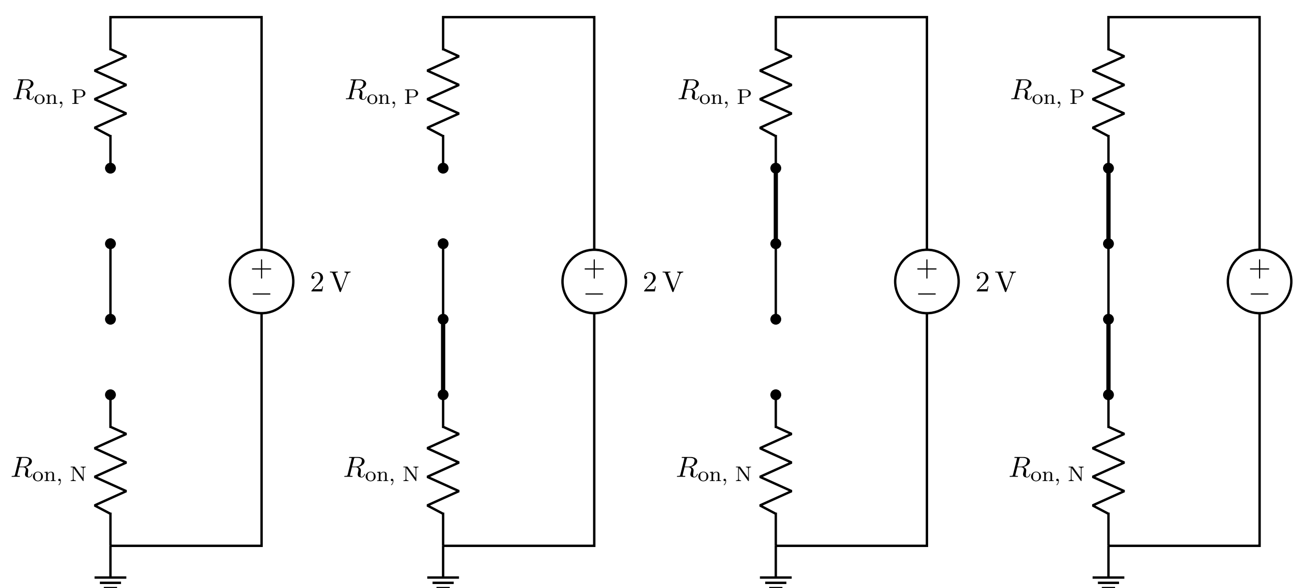


Circuit A Circuit B Circuit C

	A	B	C
Equivalent Circuit	◯	◯	◯

(c) Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**





Circuit A Circuit B Circuit C Circuit D

	A	B	C	D
Equivalent Circuit	◯	◯	◯	◯

5. Classification of Sinusoids

This HW problem can be viewed as a warm-up for a future topic in Module 2 of the course, which is going to be motivated by figuring out how to process signals recorded from the brain to decipher what a person wants to do in terms of a specific command to their robot arm. These kinds of problems are called "classification" problems. In this exercise, you will be using Python in a jupyter notebook to classify sinusoids. Sinusoids are a nice little toy universe for signals of interest that we can study. Later on in the course, we will actually be classifying speech signals in lab and understanding the techniques required to generalize this simple toy story to more practical problems. For this problem, all the ideas are from 16A and so this also serves as another gentle review for you.

The Jupyter notebook Sinusoidal_Projection_prob.ipynb will guide you through the process of performing sinusoidal projections.

Suppose you already know the true potential frequencies f_i and potential phases ϕ_i of a set of sinusoidal signals

$$S := \left\{\sin(2\pi f_i k + \phi_i), i = 1, 2, \ldots, n\right\}, \tag{5}$$

and you have some noisy samples of these true sinusoidal signals. You want to determine the true sinusoidal signal for each of these noisy samples—How would you approach the problem?

We will show in this problem that we can project noisy sinusoidal signals onto noiseless sinusoids to achieve good classification.

In the realistic world, one often doesn't have the complete waveform of a continuous function, instead oftentimes one works with *samples* of the continous function.

In our case, we generate noisy samples of the true sinusoidal signals in the following way. For each of the $num\_sinusoids$ true frequencies, each noisy sample y_i consists of N sample points sampled with a sampling rate of F_s sample rate, and corrupted by noise scaled by σ.

$$y_i(k) = \sin\big(2\pi f \cdot k/F_s\big) + \sigma \cdot Noise. \quad k = 1, 2, \ldots, N. \tag{6}$$

In our example, we will work with $num\_sinusoids = 3$.

A higher σ corresponds to more noise in our measurements.

Please complete the notebook by following the instructions given.

(a) Run the first part of the jupyter notebook to generate our noisy data points. **Use** $\sigma = 0.1, 1.0, 10.0, 100.0$ **and comment on what you observe in the plots.**

(b) **Complete part (b) of the notebook** to project noisy sinusoids onto potential true sinusoids.

Sketch the resulting 3D plot of projections qualitatively. Comment on what happens when you try the noise scalings $\sigma = 0.1, 1.0, 10.0, 100.0$**.**

(c) **Complete part (c) of the notebook** to classify the data points and calculate the number of misclassified points.

Report the number of misclassifications for $\sigma = 0.1, 1.0, 10.0, 100.0$**. Explain what happens when there is a high level of noise.** Recall that our noisy process is random so that there can be cases where there are misclassifications even in low noise.

(d) **For what qualitative regions of the noise level is it very beneficial for us to use projections?** For very low values of noise, do you have to do projections to successfully classify? What else could you have done? This question is asking you to reflect on what you have observed.

6. Existence and uniqueness of solutions to differential equations

Let's show that if any function x satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{7}$$

as well as

$$x(0) = x_0, \tag{8}$$

then it is unique: if y is *any function* that meets these two criteria then $x = y$.

In order to do this, we will first verify that a solution exists. Then we will compare it to a hypothetical alternative solution—and our goal will to be establish that these two solutions are equal.

(a) **Verify that $x_d(t) = x_0 \mathrm{e}^{\alpha t}$ satisfies (7) and (8).** (For this proof, x_d will be the "reference solution" against which alternates will be compared.)

(b) To show that this solution is in fact unique, we need to consider a hypothetical $y(t)$ that also satisfies (7) and (8).

Our goal is to show that $y(t) = x(t)$ for all $t \geq 0$. (The domain $t \geq 0$ is where we have defined the conditions (7) and (8). Outside of that domain, we don't have any constraints.)

How can we show that two things are equal? In the past, you have probably shown that two quantities or functions are equal by starting with one of them, and then manipulating the expression for it using valid substitutions and simplifications until you get the expression for the other one. However, here, we don't have an expression for $y(t)$ so that style of approach won't work.

In such cases, we basically have a couple of basic ways of showing that two things are the same.

- Take the difference of them, and somehow argue that it is 0.
- Take the ratio of them, and somehow argue that it is 1.

We will follow the ratio approach in this problem. First assume that $x_0 \neq 0$. In this case, we are free to define $z(t) = \frac{y(t)}{x_d(t)}$ since we are dividing by something other than zero.

What is $z(0)$?

(c) **Take the derivative $\frac{\mathrm{d}}{\mathrm{d}t}z(t)$ and simplify using (7) and what you know about the derivative of $x_d(t)$.**

(HINT: The quotient rule for differentiation might be helpful since a ratio is involved.)

You should see that this derivative is always 0 and hence $z(t)$ does not change. **What does that imply for y and x_d ?**

(d) At this point, we have shown uniqueness in most cases. Just one special case is left: $x_0 = 0$. The ratio technique omitted this case, because as $x_d(t) = 0$, x_d cannot be the denominator of a fraction.

To complete our proof we must show that if $x_0 = 0$, then $y(t) = 0$ for all t, and we will do so by assuming that $y(t)$ is not identically 0 for $t > 0$—that is, at some $t_0 > 0$ $y(t_0) = k \neq 0$.

From (8), we know that $y(0) = 0$. In the subsequent sub parts, we will try to work backwards in time from the point $t = t_0$ to $t = 0$ and conclude that y violates (8) if $y(t_0) \neq 0$.

Apply the change of variables $t = t_0 - \tau$ to (7) to get a new differential equation for $\widetilde{x}(\tau) = x(t_0 - \tau)$ that specifies how $\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{x}(\tau)$ must relate to $\widetilde{x}(\tau)$. This should hold for $-\infty < \tau \leq t_0$.

(e) Because the previous part resulted in a differential equation of a form for which we have already proved uniqueness for the case of nonzero initial condition, and since $\widetilde{y}(0) = y(t_0) = k \neq 0$, we know what

$\widetilde{y}(\tau)$ must be. **Write the expressions for $\widetilde{y}(\tau)$ for $\tau \in [0, t_0]$ and what that implies for $y(t)$ for $t \in [0, t_0]$.**

(f) **Evaluate $y(0)$ and argue that this is a contradiction for the specified initial condition** (8).

Consequently, such a $y(t)$ cannot exist and only the all zero solution is permitted — establishing uniqueness in this case of $x_0 = 0$ as well.

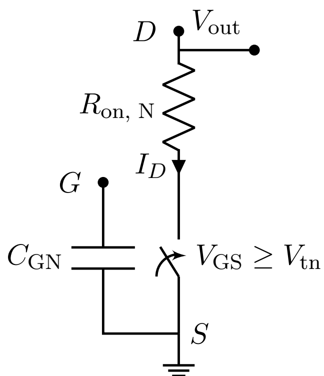
(g) Explain in your own words why it matters that solutions to these differential equations are unique.

Although we gave you lots of guidance in this problem, we hope that you can internalize this way of thinking.

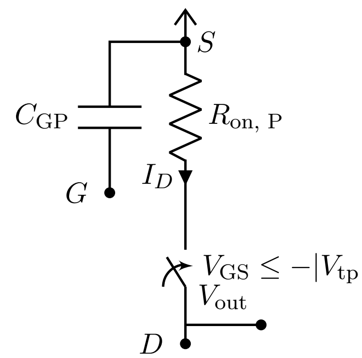
This elementary approach to proving the uniqueness of solutions to differential equations works for the kinds of linear differential equations that we will tend to encounter in EECS16B. For more complicated nonlinear differential equations, further conditions are required for uniqueness (appropriate continuity and differentiability) and proofs can be found in upper-division mathematics courses on differential equations when you study the Picard-Lindelöf theorem. (It involves looking at the magnitude of the difference of the two hypothetical solutions and showing this has to be arbitrarily small and hence zero. However, the basic elementary case we have established here can be viewed as a building block — the quotient rule gets invoked in the appropriate place, etc. The additional ingredients that are out-of-scope for lower-division courses are fixed-point theorems — which you can think of as more general siblings of the intermediate-value theorem you saw in basic calculus.)

7. Transistor Switch Model

We can improve our resistor-switch model of the transistor by adding in a gate capacitance. In this model, the gate capacitances C_{GN} and C_{GP} represent the lumped physical capacitance present on the gate node of all transistor devices. This capacitance is important as it determines the delay of a transistor logic chain.



((a)) NMOS Transistor Resistor-switch-capacitor model



((b)) PMOS Transistor Resistor-switch-capacitor model. Note we have drawn this so that it aligns with the inverter.

You have two CMOS inverters made from NMOS and PMOS devices. Both NMOS and PMOS devices have an "on resistance" of $R_{\text{on, N}} = R_{\text{on, P}} = 1\ \text{k}\Omega$, and each has a gate capacitance (input capacitance) of $C_{\text{GN}} = C_{\text{GP}} = 1\,\text{fF}$ (fF = femto-Farads = 1×10^{-15} F). We assume the "off resistance" (the resistance when the transistor is off) is infinite (i.e., the transistor acts as an open circuit when off). The supply voltage V_{DD} is 1V. The two inverters are connected in series, with the output of the first inverter driving the input of the second inverter (Figure 9).

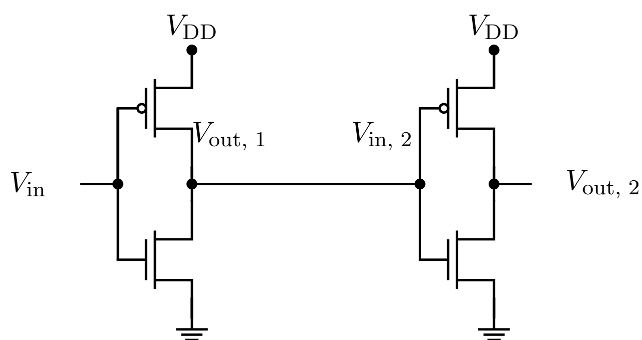


Figure 9: CMOS Inverter chain

(a) Assume the input to the first inverter has been low ($V_{\text{in}} = 0$ V) for a long time, and then switches at time $t = 0$ to high ($V_{\text{in}} = V_{\text{DD}}$). **Draw a simple RC circuit and write a differential equation describing the output voltage of the first inverter ($V_{\text{out, 1}}$) for time $t \geq 0$.** Don't forget that the second inverter is "loading" the output of the first inverter — you need to think about both of them.

(b) Given the initial conditions in part (a), **solve for $V_{\text{out, 1}}(t)$.**

(c) **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the initial slope, (3) the asymptotic value, and (4) the time that it takes for the voltage to decay to roughly 1/3 of its initial value.**

(d) A long time later, the input to the first inverter switches low again.
Solve for $V_{\text{out, 1}}(t)$.
Sketch the output voltage of the first inverter ($V_{\text{out, 1}}$), showing clearly (1) the initial value, (2) the initial slope, and (3) the asymptotic value.

8. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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