EECS 16B Designing Information Devices and Systems II
Fall 2021 UC Berkeley

Homework 02

This homework is due on Friday, September 10, 2021, at 11:59PM. Self-grades and HW Resubmissions are due on Tuesday, September 14, 2021, at 11:59PM.

1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here is a link to the note that you need to read for this week: Note 2.

(a) **How do we deal with piecewise constant inputs as introduced in the notes?** Just answer briefly with the key idea.

2. Uniqueness Counterexample

This problem explores an example of a differential equation that does not have a unique solution. The purpose is to show that uniqueness cannot always be assumed. There is a reason we are making you prove uniqueness to trust solutions.

Along the way, this problem will also show you a heuristic way to guess the solutions to differential equations that is often called "separation of variables." The advantage of the separation of variables technique is that it can often be helpful in systematically coming up with guesses for nonlinear differential equations. However, as with any technique for guessing, it is not a proof and the guess definitely needs to be checked and uniqueness verified before proceeding.

The idea of separation-of-variables is to pretend that $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$ is a ratio of quantities rather than what it is — a shorthand for taking the derivative of the function $x(\cdot)$ with respect to its single argument, and then writing the result in terms of the free variable "t" for that argument. This little bit of make-believe (sometimes euphemistically called "an abuse of notation") allows one the freedom to do calculations.

To demonstrate, let's do this for a case where we know the correct solution: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t)$. This is how a separation-of-variables approach would try to get a guess:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) \tag{1}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x \tag{2}$$

$$\frac{\mathrm{d}x}{x} = \lambda\,\mathrm{d}t \qquad \text{separating variables to sides} \tag{3}$$

$$\int \frac{\mathrm{d}x}{x} = \int \lambda\,\mathrm{d}t \qquad \text{integrating both sides} \tag{4}$$

$$\ln(x) + C_1 = \lambda t + C_2 \tag{5}$$

$$x(t) = K\mathrm{e}^{\lambda t} \qquad \text{exponentiating both sides and folding constants} \tag{6}$$

With the above guess obtained, $x(t) = K\mathrm{e}^{\lambda t}$ can be plugged in and seen to solve the original differential equation because the steps above are vaguely reversible, if a bit hallucinatory in nature. Then of course, a uniqueness proof is required, but you did that in the previous homework.

To see why this technique is a bit fraught, we will consider the following nonlinear differential equation involving a third root. (If we had more time in making this problem, we would have showed you how this sort of differential equation can arise from a toy physical setting of a inverted pyramidal container that had $x(t)$ liters of water in it, where the rate of water being poured in is proportional to the height of the water $x^{\frac{1}{3}}$. This fractional power arises since volume is a cubic quantity while the water is being poured in at a rate governed by a one-dimensional quantity of length. Similar equations can arise in microfluidic dynamics.)

Anyway, consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x^{\frac{1}{3}} \tag{7}$$

with the initial condition

$$x(0) = 0. \tag{8}$$

Let's take the "separation-of-variables pill" and see what trip it takes us on:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x^{\frac{1}{3}} \tag{9}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x^{\frac{1}{3}} \tag{10}$$

$$x^{-\frac{1}{3}}\,\mathrm{d}x = \alpha\,\mathrm{d}t \tag{11}$$

$$\int x^{-\frac{1}{3}}\,\mathrm{d}x = \int \alpha\,\mathrm{d}t \tag{12}$$

$$\frac{3}{2}x^{\frac{2}{3}} + C_1 = \alpha t + C_2 \tag{13}$$

$$x = \left(\frac{2}{3}\alpha t + C_3\right)^{\frac{3}{2}} \tag{14}$$

That didn't seem like too wild a ride. Let's see what rabbit hole we have actually landed in.

(a) Given our separation-of-variables based calculation, let us guess a solution of the form

$$x(t) = \left(\frac{2}{3}\alpha t + c\right)^{\frac{3}{2}} \tag{15}$$

Show that this is a solution to the differential equation (7), and find the c that satisfies the initial condition.

(HINT: You'll need to use the power rule and chain rule.)

(b) Let us guess a second solution:

$$x(t) = 0 \tag{16}$$

Show that this new guess also satisfies (7), and the initial condition ($x(0) = 0$).

(c) **Show that any solution of the form**

$$x(t) = \begin{cases} 0, & \text{if } t < t_0 \\ \left(\frac{2}{3}\alpha\left(t - t_0\right)\right)^{\frac{3}{2}}, & \text{if } t \geq t_0 \end{cases} \tag{17}$$

also satisfies (7) and the initial condition $x(0) = 0$, for any $t_0 > 0$.

So this actually has *infinitely* many solutions.

(d) A known (not by you yet, but by the mathematical community) sufficient condition for the uniqueness of solutions to differential equations of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t))$ is that the function $f(x)$ be continuously differentiable (i.e. $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ is a continuous function of x) with a bounded derivative $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ at the initial condition $x(0)$ and everywhere that the solution $x(t)$ purports to go. (You will understand the importance of this condition and where it comes from better when we are in Module 2 of 16B. We are not going to prove it.)

Does this differential equation problem satisfy this condition that would let us trust guessing and checking?

(e) The separation-of-variables trip can be considered like a dream sequence if you'd like — it has an internal logic to it, just not a logic that you can rely upon in the real world. In this particular case, there is actually a little bit of a warning in the separation-of-variables trip argument. **Explain why** (11) **might be a bit problematic in this case.**

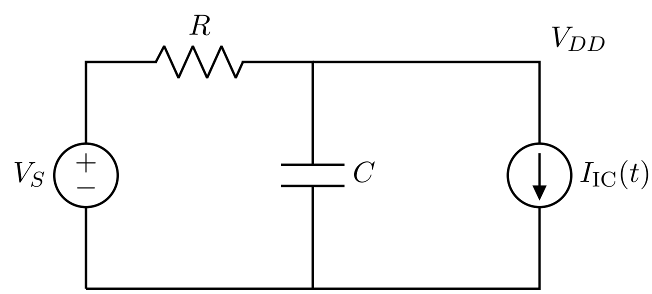
(HINT: When is it not permissible to divide both sides of an equation by the same thing?)

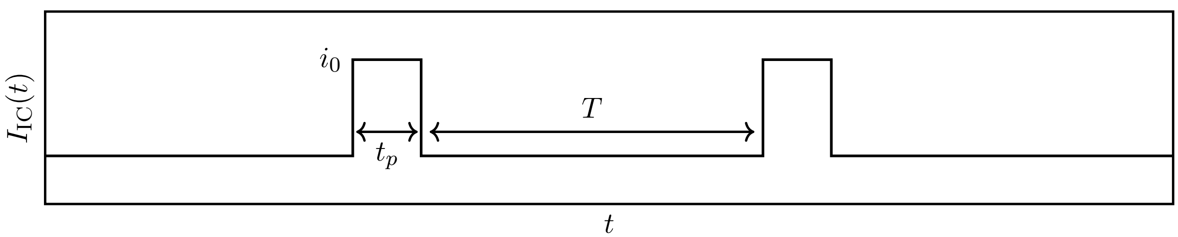
(f) To see that separation of variables can actually give you reasonable guesses even for equations with interesting solutions, try to use it to solve the following differential equation $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{1}{2}x^2$ for initial condition $x(0) = 1$. **Sketch your solution. Describe anything interesting that happens. Does this problem satisfy the uniqueness conditions for** $t \in [0, 1]$**?**

3. IC Power Supply

Digital integrated circuits (ICs) often have very non-uniform current requirements which can cause voltage noise on the supply lines. If one IC is adding a lot of noise to the supply line, it can affect the performance of other ICs that use the same power supply, which can hinder performance of the entire device. For this reason, it is important to take measures to mitigate, or "smooth out", the power supply noise that each IC creates. A common way of doing this is to add a "supply capacitor" between each IC and the power supply. (If you look at a circuit board, and the supply capacitor is the small capacitor next to each IC.)

Here's a simple model for a power supply and digital circuit:





The current source is modeling the "spiky," non-uniform nature of digital circuit current consumption. 'The resistor represents the sum of the source resistance of the supply and any wiring resistance between the supply and the load.

The capacitor is added to try to minimize the noise on V_{DD}. Assume that $V_S = 3\,\text{V}$, $R = 1\,\Omega$, $i_0 = 1\,\text{A}$, $T = 10\,\text{ns}$, and $t_p = 1\,\text{ns}$.

(a) **Sketch the voltage V_{DD} vs. time for two periods T assuming that $C = 0$.**

(b) **Give expressions for and sketch the voltage V_{DD} vs. time for two periods T for each of three different capacitor values for C: $1\,\text{pF}$, $1\,\text{nF}$, $1\,\mu\text{F}$. ($1\,\text{pF} = 1 \times 10^{-12}\,\text{F}$, $1\,\text{nF} = 1 \times 10^{-9}\,\text{F}$, $1\,\mu\text{F} = 1 \times 10^{-6}\,\text{F}$).** For this part, to find the initial condition for V_{DD}, feel free to assume that for a very long time, $I_{IC} = 0$.

(c) Launch the attached Jupyter notebook to interact with a simulated version of this IC power supply. Try to simulate the scenarios outlined in the previous parts. For one of these scenarios, keep the RC time constant fixed, but vary the relative value of R vs. C (e.g. compare $R = \texttt{1}, C = \texttt{2e-9}$ to the case where $R = \texttt{2}, C = \texttt{1e-9}$). **Is it better to have a lower R or lower C value for a fixed RC time constant when attempting to minimize supply noise? Give an intuitive explanation for why this might be the case.**

Be sure to play with the y limits on the graph as well as how long the simulation runs to best understand what is going on here.

4. Simple scalar differential equations driven by an input

In class, you learned that the solution for $t \geq 0$ to the simple scalar first-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) \tag{18}$$

with initial condition

$$x(t = 0) = x_0 \tag{19}$$

is given for $t \geq 0$ by

$$x(t) = x_0 \mathrm{e}^{\lambda t}. \tag{20}$$

In an earlier homework, you proved that these solutions are *unique*– that is, that $x(t)$ of the form in (20) are the only possible solutions to the equation (18) with the specified initial condition (19).

In this question, we will analyze differential equations with inputs and prove that their solutions are unique.

In particular, we consider the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) + u(t) \tag{21}$$

where $u(t)$ is a known function of time from $t = 0$ onwards.

(a) Suppose that you are given an $x_g(t)$ that satisfies both (19) and (21) for $t \geq 0$.
Show that if $y(t)$ also satisfies (19) and (21) for $t \geq 0$, then it must be that $y(t) = x_g(t)$ for all $t \geq 0$.
(HINT: You already used ratios in an earlier HW to prove that two things were necessarily equal. This time, you might want to use differences. Be sure to leverage what you already proved earlier instead of having to redo all that work.)

(b) Suppose that the given $u(t)$ starts at $t = 0$ (it is zero before that) and is a nicely integrable function (feel free to assume bounded and continuously differentiably with bounded derivative — whatever conditions you assumed in your calculus course when considering integration and the fundamental theorem of calculus). Let

$$x_c(t) = x_0 \mathrm{e}^{\lambda t} + \int_0^t u(\tau) \mathrm{e}^{\lambda(t-\tau)} \, \mathrm{d}\tau \tag{22}$$

for $t \geq 0$.
Show that the $x_c(t)$ defined in (22) indeed satisfies (21) and (19).
Note: the τ here in (22) is just a dummy variable of integration. We could have used any letter for that local variable. We just used τ because it visually reminds us of t while also looking different. If you think they look too similar in your handwriting, feel free to change the dummy variable of integration to another symbol of your choice.
(HINT: Remember the fundamental theorem of calculus that you proved in your calculus class and manipulate the expression in (22) to get it into a form where you can apply it along with other basic calculus rules.)

(c) **Use the previous part to get an explicit expression for $x_c(t)$ for $t \geq 0$ when $u(t) = \mathrm{e}^{st}$ for some constant s, when $s \neq \lambda$ and $t \geq 0$.**

(d) **Similarly, what is $x_c(t)$ for $t \geq 0$ when $u(t) = \mathrm{e}^{\lambda t}$ for $t \geq 0$.**
(HINT: Don't worry if this seems too easy.)

5. Op-Amp Stability

In this question we will revisit the basic op-amp model that was introduced in EECS 16A and we will add a capacitance C_{out} to make the model more realistic (refer to figure 1). Now that we have the tools to do so, we will study the behavior of the op-amp in positive and negative feedback (refer to figure 2). Furthermore, we will begin looking at the integrator circuit (refer to figure 3) to see how a capacitor in the negative feedback can behave. In the next homework, you will see why it ends up being close to an integrator.

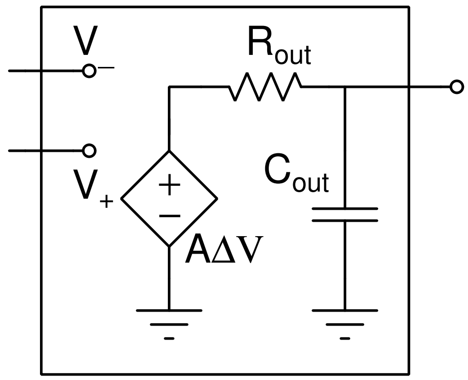
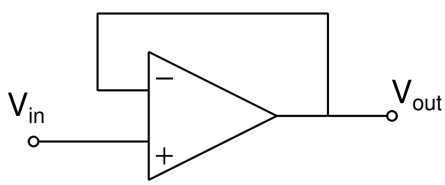
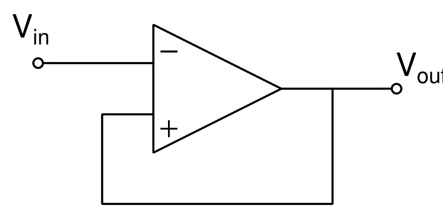


Figure 1: Op-amp model: $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback



(b) Op-amp in positive feedback

Figure 2: Op-amp in positive and negative feedback

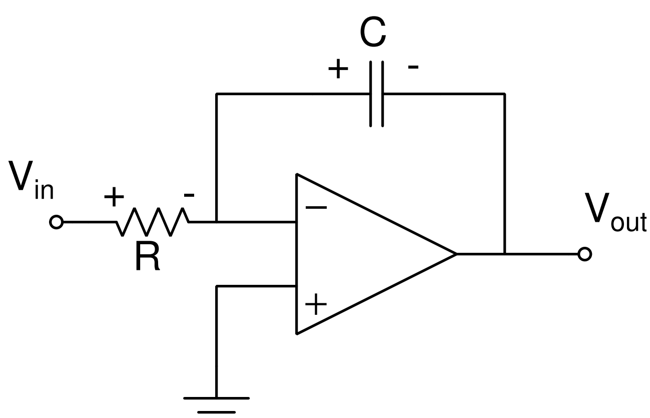


Figure 3: Integrator circuit

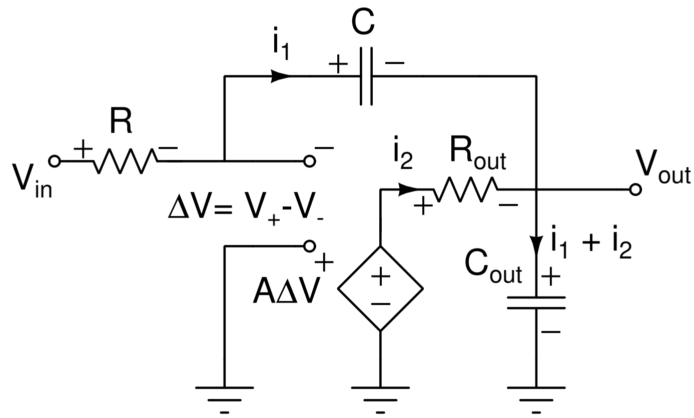


Figure 4: Integrator circuit with Op-amp model

(a) Using the op-amp model in figure 1 and the buffer in negative-feedback configuration in figure 2a, **draw a combined circuit**. Remember that $\Delta V = V_+ - V_-$, the voltage difference between the positive and negative labeled input terminals of the op-amp.

(HINT: Look at figure 4 to see how this was done for the integrator. That might help.)

Note: here, we have used the equivalent model for the op-amp gain. In more advanced analog circuits courses, it is traditional to use a controlled current source with a resistor in parallel instead.

(b) Let's look at the op-amp in negative feedback. From our discussions in EECS 16A, we know that the buffer in figure 2a should work with $V_{\text{out}} \approx V_{\text{in}}$ by the golden rules. **Write a differential-equation for V_{out} by replacing the op-amp with the given model and show what the solution will be as a function of time for a static V_{in}. What does it converge to as $t \to \infty$?** Note: We assume the gain $A > 1$ for all parts of the question. For this part, you can assume the initial condition $V_{\text{out}}(0) = 0$.

(c) Next, let's look at the op-amp in positive feedback. We know that the configuration given in figure 2b is unstable and V_{out} will just rail. **Again, using the op-amp model in figure 1, show that V_{out} does not converge and hence the output will rail. For positive DC input $V_{\text{in}} > 0$, will V_{out} rail to the positive or negative side? Explain.** For this part, you can assume the initial condition $V_{\text{out}}(0) = 0$.

(d) For an ideal op-amp, we can assume that it has an infinite gain, i.e., $A \to \infty$. **Under these assumptions, show that the op-amp in negative feedback behaves as an ideal buffer, i.e., $V_{\text{out}} = V_{\text{in}}$.**

(e) Let's extend our analysis to the integrator circuit shown in figure 3. Simplifying all the equations, we get a system of differential equations in two variables V_C and V_{out}, where V_C and V_{out} are the voltage drops across the capacitors C and C_{out}. **Fill in the missing term in the following matrix differential equation.**

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_{\text{out}}(t) \\ V_C(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}}\right) & -\left(\frac{1}{RC_{\text{out}}} + \frac{A}{R_{\text{out}}C_{\text{out}}}\right) \\ -\frac{1}{RC} & ? \end{bmatrix} \begin{bmatrix} V_{\text{out}}(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_{\text{out}}} \\ \frac{1}{RC} \end{bmatrix} V_{\text{in}}(t) \tag{23}$$

(HINT: We picked an easier term to hide. You don't have to write out all the equations and do a lot of algebra to figure out what the missing term is.)

6. Successive Approximation Register Analog-to-Digital Converter (SAR ADC)

An analog-to-digital converter (ADC) is a circuit for converting an analog voltage into digital approximate representation of that voltage. One commonly used circuit architecture for analog-to-digital converters is the Successive Approximation Register ADC (SAR ADC), which you will see in Lab 3. An N-bit SAR ADC converts an input analog voltage to an N-bit binary string between 0 and $2^N - 1$ if we viewed that binary string as representing an integer in binary.

The SAR ADC does this by following one of the key themes in 16B, reducing a problem to something that we already know how to do. In this case, the two ingredients are the DAC (digital to analog converter) that we saw in HW 1; and the idea of a binary search tree that you saw in 61A. As you remember from 61A, the key operation required for a binary search is being able to do a less-than-greater-than comparison. Fortunately, we have a circuit element from 16A that lets us do that, namely a comparator.

Explicitly, the SAR ADC implements the binary search algorithm by feeding trial codes into a **DAC**, like the one we analyzed in Homework 1, to generate voltages and comparing the resulting analog voltage with the analog input voltage using a **comparator**. It then uses feedback (**SAR logic**) to adjust the DAC output voltage to get as close as possible to the input analog voltage. The algorithm starts by setting the most significant bit (MSB), which is the bit with the largest binary weight (i.e. furthest to the left in a traditional binary number), and then moving on to set the next bit.

If this is not clear to you, think about how you would play 20 questions to guess an integer between 0 and $2^{20} - 1 \approx 1000000$. The SAR ADC does that, with the comparator playing the role of the question-answering person.

Here is an illustration of the algorithm.

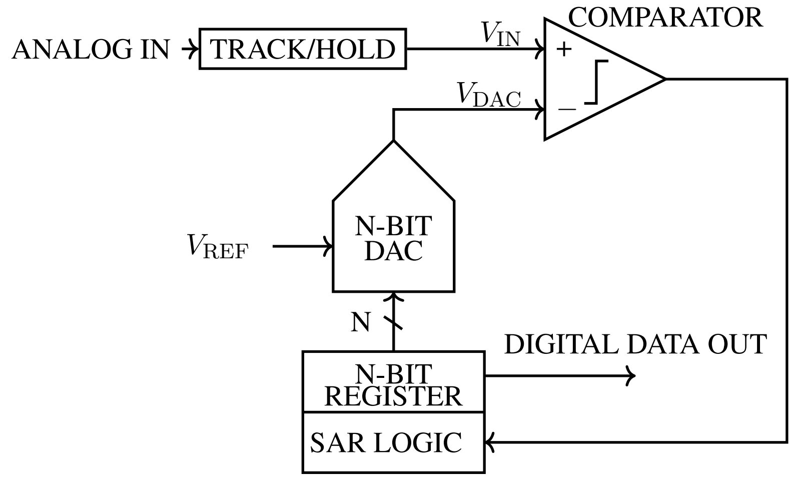


Figure 5: SAR ADC circuit diagram

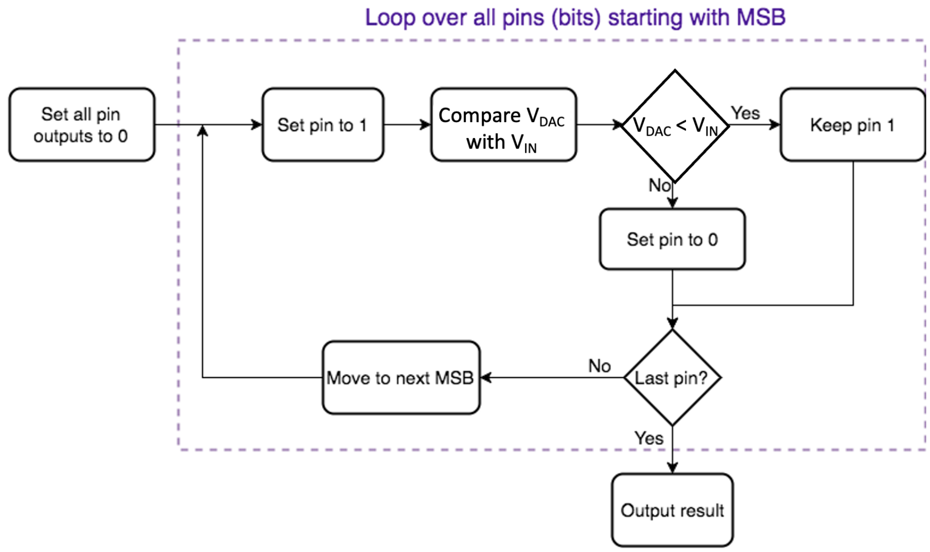


Figure 6: Flow chart of SAR ADC algorithm: 'pin' refers to each bit in the binary string

The comparator in Figure 5 outputs logic high (1) when $V_{\mathrm{IN}} \geq V_{\mathrm{DAC}}$ and logic low (0) when $V_{\mathrm{IN}} < V_{\mathrm{DAC}}$. Let's look at a 3-bit SAR ADC as an example. The voltage transfer curve of the 3-bit SAR ADC is shown in Figure 7. For $V_{\mathrm{IN}} = 0.3V_{\mathrm{REF}}$, the operation of SAR ADC is shown in Figure 8.

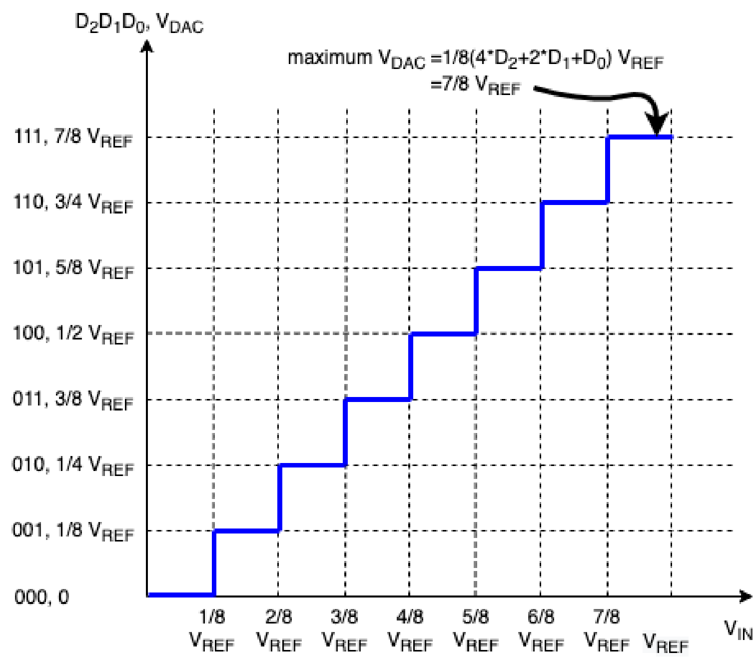


Figure 7: Voltage transfer curve of 3-bit SAR ADC: $D_2D_1D_0$ is the output digital code of the ADC. Notice that the maximum voltage of the V_{DAC} is $\frac{2^N-1}{2^N}V_{\mathrm{REF}}$ where N is the number of bits ($N = 3$ in this case)

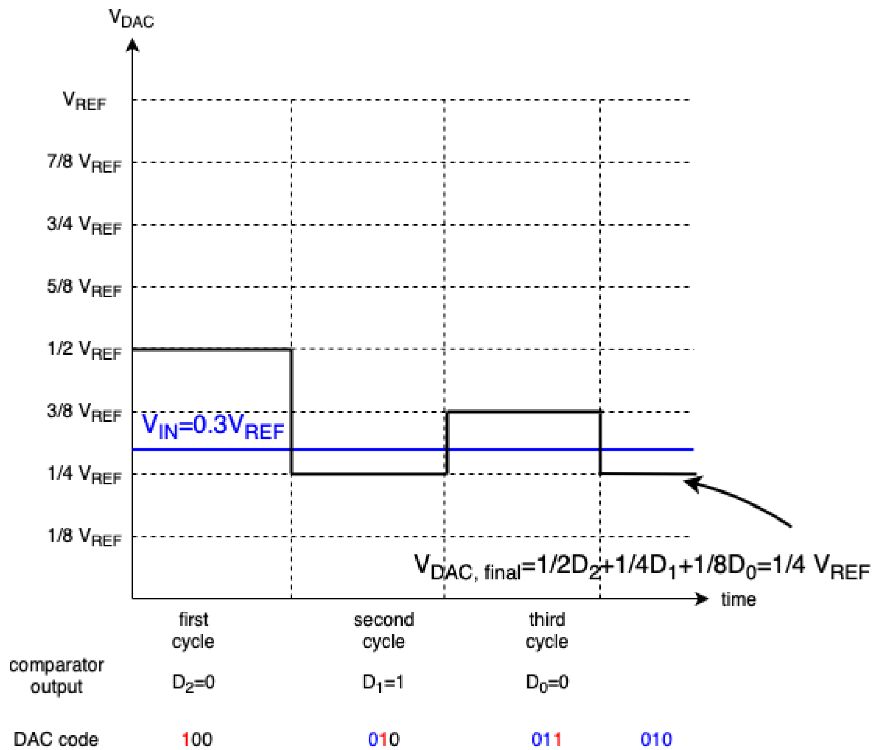


Figure 8: Timing diagram of SAR output code. Red digit in DAC code is the bit being decided in the cycle; blue digits are the bits determined by the previous conversion cycles.

From the timing diagram in Figure 8, it can be seen that the final digital output code is 010 which represents the closest DAC output voltage $\left(\frac{1}{4}V_{\text{REF}}\right)$ to the input analog voltage $V_{\text{IN}} = 0.3V_{\text{REF}}$ with a 3 bit binary representation.

We will now analyze the operation of a 4-bit SAR ADC with the input voltage range between $0\,\text{V}$ and $5\,\text{V}$ and output code range between 0000 (0) and 1111 (15). In other words, V_{REF} is $5\,\text{V}$ and $0\,\text{V} < V_{\text{in}} < 5\,\text{V}$ in Figure 5.

(a) **What is the corresponding input voltage range of the ADC for output code 0000?** *(HINT: Try to draw the transfer curve for 4-bit SAR ADC, similar to that for the 3-bit SAR ADC in Figure 7. What values of* V_{IN} *will map to 0000* $(V_{\text{DAC}} = 0\,\text{V})$ *in this case?)*

(b) If the input analog voltage is $V_{\text{IN}} = 3\,\text{V}$, then answer the following questions:

  i. **What is the ratio between the input voltage V_{IN} and the maximum input voltage $V_{\text{REF}} = 5\,\text{V}$? What should be the output code of the SAR ADC?**

  ii. Operation of 4-bit SAR ADC (see Figure 8 as reference)

    i. **Plot the output voltage of the DAC in the timing diagram in Figure 9.**

    ii. **Fill out the output of the comparator in each conversion cycle.**

    iii. **Fill out the ADC output code ($D_3D_2D_1D_0$) in each conversion cycle.**

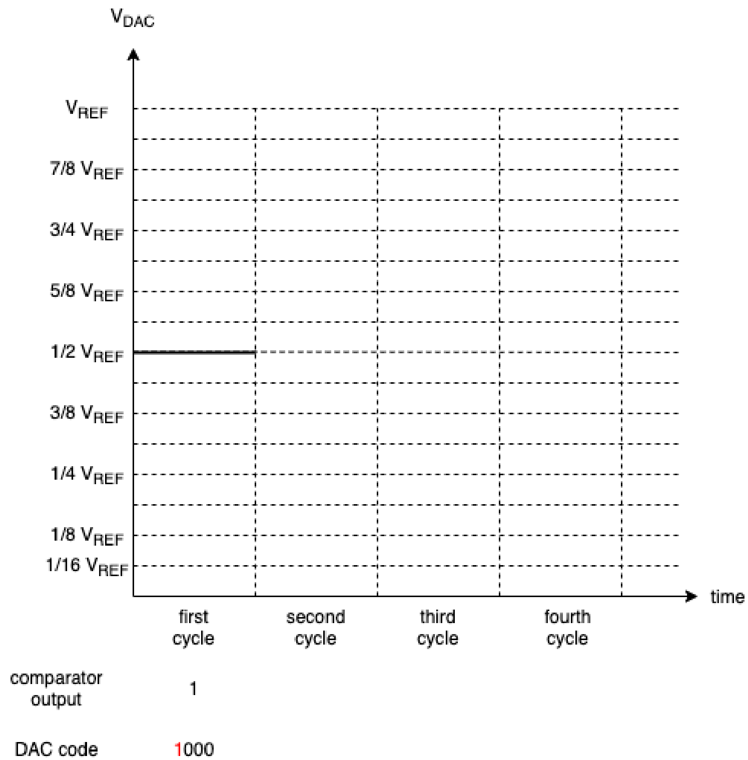


Figure 9: SAR ADC V_{DAC} timing diagram

iii. **What is the final output voltage of the DAC at the end of the conversion? Is it the same as the input voltage? If not, what is the voltage difference between the final output voltage of the DAC V_{DAC} and the input voltage V_{IN}?**

7. Do you need to take the exam remotely?

Please fill out this Google Form if you are a student who is not physically near Berkeley (more than 150 miles away) or if you are a student who has University-approved reasons (like an exception to the vaccine mandate) to request a remote alternative to the exam(s) with strict, heavy video+audio proctoring that will occur at the same exact time as the in-person exam(s). Refer to Piazza post @254 for more details. **If you do not need to take the exam remotely, just write "No" for this problem. You shouldn't fill out the form.**

8. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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