EECS 16B Fall 2021 — Designing Information Devices and Systems II, UC Berkeley

Homework 03

This homework is due on Friday, September 17, 2021, at 11:59PM. Self-grades and HW Resubmissions are due on Tuesday, September 21, 2021, at 11:59PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 3 Note 4

   (a) When writing differential equations, what should you choose as your state variables?

   (b) When is a matrix diagonalizable?

   (c) How do we solve differential equations when a derivative of a state variable depends on other state variables besides itself? (This is called coupling between states.)

   (d) When inductors feature as an element in circuit, what must be taken as a state variable?

2. Taking limits to understand differential equations with inputs

Working through this question will help you understand better differential equations with inputs. This problem also provides a vehicle to review some more relevant concepts from calculus. The homework problem here builds directly on the Discussion 2B you just had, and so if for whatever reason you did not follow discussion, you need to first make sure you understand that.

The goal in this problem is to see how you can use what you already know about piecewise constant inputs to take limits and use those limits to guess what the solution is to the scalar differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t). \tag{1}$$

You've already proven in another homework problem that the solution to this differential equation is given by an appropriate integral and that it is unique. The goal here is to see how you could have come up with that guess.

Discussion recap to make this problem self-contained:

To come up with a guess, we started with a piecewise constant $u(t)$; we already had the tools to solve this since it was a natural extension of the kind of analysis we did in Discussion 1B, part c, where we chained together 2 intervals' results to form a continuous curve.

We supposed that our input $u(t)$ of interest is piecewise constant over durations of width Δ. In other words:

$$u(t) = u(i\Delta) = u[i] \text{ if } t \in [i\Delta, (i+1)\Delta) \equiv i\Delta \leq t < (i+1)\Delta. \tag{2}$$

To stay consistent, we will use the notation

$$x_d[i] = x(i\Delta).$$

Here, the square brackets are designed to remind you of typical array indexing since that is what is essentially going on here. The $\ldots, u[i], u[i+1], \ldots$ are basically entries in an array or stream indexed by i.

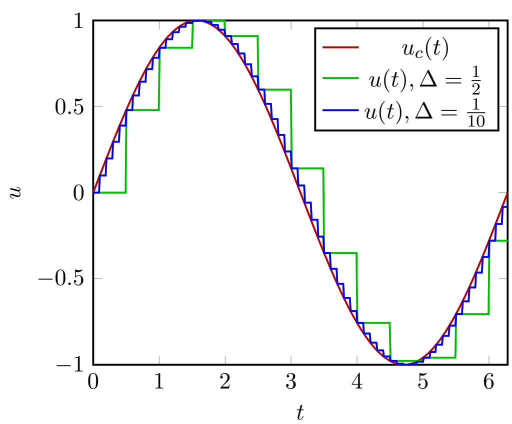


Figure 1: An example of a discrete input where the limit as the time-step Δ goes to 0 approaches a continuous function. The red line, the original signal $u_c(t) = \sin(t)$, is traced almost exactly by the blue line, which has a small time-step, and not nearly as well by the green line, which has a large time-step.

The first step to analyzing this system was to discover its behavior across a single time-step where the input stays constant, since we already know how to solve these kinds of systems.

We first looked at one step and having assumed that we knew the value of $x(i\Delta) = x_d[i]$, set out to compute $x_d[i+1] = x((i+1)\Delta)$.

Here is a solution to this system, which may help with visual intuition:

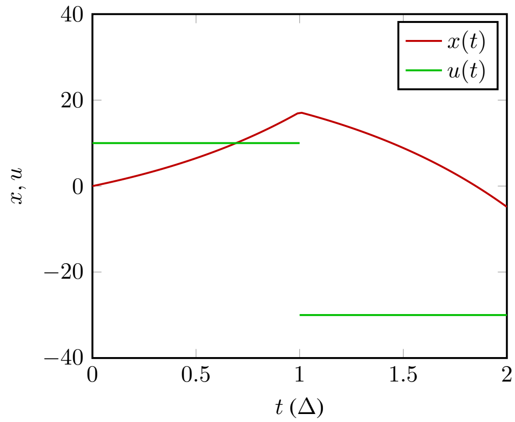


Figure 2: An example of a solution to this diff. eq. system. In this case $\lambda = 1, u[0] = 10, u[1] = -30$.

Looking at the first-order differential equation for $t \in [i\Delta, (i+1)\Delta)$, we saw that it was

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t) = \lambda x(t) + u[i]. \tag{3}$$

Grinding it out, we got the solution:

$$x(t) = \alpha \mathrm{e}^{\lambda(t-i\Delta)} + \beta = \left(x_d[i] + \frac{u[i]}{\lambda}\right)\mathrm{e}^{\lambda(t-i\Delta)} - \frac{u[i]}{\lambda} \tag{4}$$

$$= \mathrm{e}^{\lambda(t-i\Delta)} x_d[i] + \frac{\mathrm{e}^{\lambda(t-i\Delta)} - 1}{\lambda} u[i] \tag{5}$$

The reason we simplified in this manner is because we want to split the value of $x(t)$ into the effect of the initial condition $x_d[i]$, and the input $u[i]$.

Now since $x(t)$ is continuous across all t, $x_d[i+1] = x((i+1)\Delta)$. Thus

$$x_d[i+1] = x((i+1)\Delta) = \mathrm{e}^{\lambda((i+1)\Delta - i\Delta)} x_d[i] + \frac{\mathrm{e}^{\lambda((i+1)\Delta - i\Delta)} - 1}{\lambda} u[i] \tag{6}$$

$$= \mathrm{e}^{\lambda\Delta} x_d[i] + \frac{\mathrm{e}^{\lambda\Delta} - 1}{\lambda} u[i]. \tag{7}$$

This is the quantity we wanted.

Our next step was to unroll the implicit recursion we derived to write $x_d[i+1]$ as a sum that involves $x_d[0]$ and the $u[j]$ for $j = 0, 1, \ldots, i$.

This was reminiscent of the Segway problem from 16A and HW00 and to see the link, we just considered the discrete-time system in a simpler form

$$x_d[i+1] = a x_d[i] + b u[i] \tag{8}$$

Working through a few steps, we saw the pattern

$$x_d[i] = a^i x_d[0] + b \sum_{j=0}^{i-1} a^{i-1-j} u[j]. \tag{9}$$

and ran it for one more step to see if it works:

$$x_d[i+1] = a x_d[i] + b u[i] = a\left(a^i x_d[0] + b \sum_{j=0}^{i-1} a^{i-1-j} u[j]\right) + b u[i] \tag{10}$$

$$= a^{i+1} x_d[0] + b\left(\sum_{j=0}^{i-1} a^{i-j} u[j]\right) + b u[i] \tag{11}$$

$$= a^{i+1} x_d[0] + b\left(u[i] + \sum_{j=0}^{i-1} a^{i-j} u[j]\right) \tag{12}$$

$$= a^{i+1} x_d[0] + b \sum_{j=0}^{i} a^{i-j} u[j] \tag{13}$$

We then started on the path towards taking limits. To do that, we needed to establish a correspondence between an arbitrary continuous time t and the discrete i interval that corresponds to it. We saw that

$$i = \left\lfloor \frac{t}{\Delta} \right\rfloor \tag{14}$$

is the discrete time index i that corresponds to the time t in real time, because it is the only i satisfying $t \in [i\Delta, (i+1)\Delta)$.

We then used the fact that $x(t)$ is continuous (after all, it has a derivative) to give an approximate expression for $x(t)$ for any t, in terms of $x_d[0] = x(0)$ and the inputs $u[j]$. This let us ignore what happens inside the intervals and just look at the endpoints so that we say: $x(t) \approx x\left(\Delta \left\lfloor \frac{t}{\Delta} \right\rfloor\right) = x_d\left[\left\lfloor \frac{t}{\Delta} \right\rfloor\right]$.

Grinding through the substitutions, we saw that

$$x(t) \approx \left(\mathrm{e}^{\lambda\Delta}\right)^{\left\lfloor \frac{t}{\Delta} \right\rfloor} x_d[0] + \frac{\mathrm{e}^{\lambda\Delta} - 1}{\lambda} \sum_{j=0}^{\left\lfloor \frac{t}{\Delta} \right\rfloor - 1} (\mathrm{e}^{\lambda\Delta})^{\left\lfloor \frac{t}{\Delta} \right\rfloor - 1 - j} u[j]. \tag{15}$$

At this point, we are all set up to start the process of taking the limit of $\Delta \to 0$. This is what this homework problem is about.

(a) We first need to relate the $u[i]$ to a continuous-time input. Suppose that the $u[i]$ is actually a sample of a desired input $u_c(t)$ in continuous time. Namely, suppose that $u[i] = u_c(i\Delta)$.

To clarify, $u(t)$ is a piecewise constant function; $u[i]$ is the discrete input that constructs $u(t)$; and $u_c(t)$ is the underlying input $u[i]$ is sampled from.

When we take the limit $\Delta \to 0$, the goal is is to find an expression for $x(t)$ in terms of $u_c(t)$ and the initial condition $x(0)$. To this end, **start by substituting an appropriate value of u_c for $u[j]$ in the**

result eq. (15) from the discussion. (Note: don't take any limits in this part of the problem; just do the substitution.)

(b) We want to take the limit $\Delta \to 0$ of our (discrete-time) expression and thus get a continuous-time function, but right now our discrete-time expression itself is pretty complicated. When faced with such a situation, we need to simplify it by making some approximations which become exact in the limit.

Further approximate the previous expression by considering the following two estimates:

i. Let $n = \lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$ where needed and treat $\Delta \approx \frac{t}{n}$. This is a meaningful approximation when we think about n large enough.

ii. Treat $\frac{e^{\lambda\Delta}-1}{\lambda} \approx \Delta$. This is a meaningful approximation when we think about Δ small enough. One can derive this estimate by using Taylor's theorem from calculus, but it's not required here.

(*Hint*: Use the first estimate to get rid of "floor" terms, then use both estimates to simplify further. In simplifications, try to pull outside of sums anything that does not need to be inside the sum.)

(c) Here's the main payoff! We now obtain a continuous-time expression for $x(t)$, completing the transition into continuous-time. **Take the limit of $x(t)$ as $\Delta \to 0$ or equivalently as $n \to \infty$. What is the expression you get for $x(t)$?**

Recall that the definite integral is defined from Riemann sums as

$$\int_0^t f(\tau)\,\mathrm{d}\tau = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(\tau_j^*)\Delta_n \tag{16}$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t$, $\tau_j^* \in [\tau_{j-1}, \tau_j]$, and $\Delta_n = \tau_j - \tau_{j-1}$. The Δ_n is the length of the base of the rectangles and the $f(\tau_j^*)$ are the heights. As n goes to infinity, the rectangles get skinnier and skinnier, but there are more and more of them.

(*Hint*: You're going to want to massage things in that direction.)

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. Being able to grind through complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice.

3. Tracking Terry

Terry is a mischievous child, and his mother is interested in tracking him.

(a) Terry texts his current location as a vector $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but there is a problem! These coordinates are *not* in the standard basis, but rather in the basis $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. That is to say that the first number 2 above is how many multiples of $\vec{v}_1$ to use and the second number 3 is how many multiples of $\vec{v}_2$ to use in computing his actual location. Here, both $\vec{v}_1$ and $\vec{v}_2$ are vectors in the standard basis.
Let Terry's location in the standard basis be $\vec{x}$. Write $\vec{x}$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

(b) Terry's friend tells you that Terry's location in the standard basis is $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Using this along with the previous info that Terry's location in the V basis is $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, **is it possible to determine the basis vectors $\vec{v}_1, \vec{v}_2$ Terry is using. If it is impossible to do so, explain why.**
(HINT: How many unknowns do you have? How many equations?)

(c) Terry's basis vectors $\vec{v}_1, \vec{v}_2$ get leaked to his mom on accident, so she knows they are

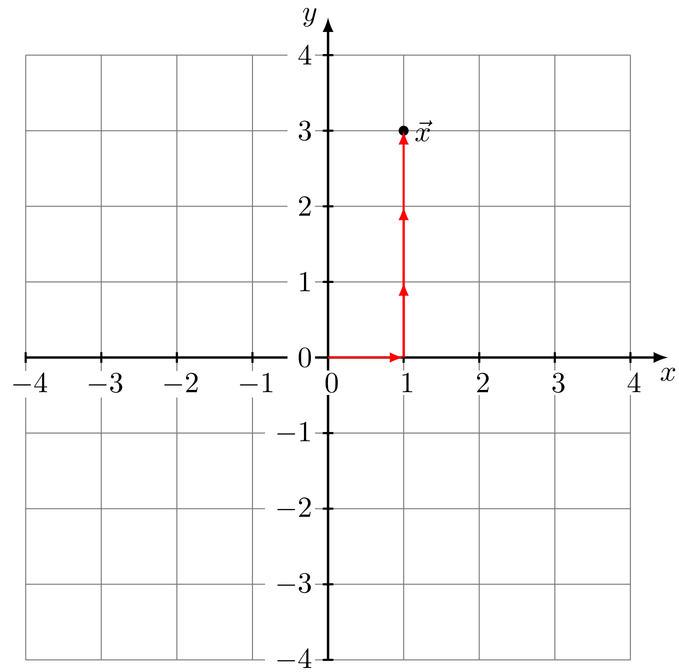
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{17}$$

To hide his location, Terry wants to switch to a new coordinate system with the basis vectors

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{18}$$

In order to do this, he needs a way to convert coordinates from the V basis to the P basis. Thus, **find the matrix T such that if $\vec{x}_v$ is a location expressed in V coordinates and $\vec{x}_p$ is the same location expressed in P coordinates, then $\vec{x}_p = T\vec{x}_v$.**

(d) Terry now wants to make a map and route to where he currently is, $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. **For both the P and V bases from part (c), illustrate the sum of scaled basis vectors that are necessary to go from the origin to $\vec{x}$.** An example is shown below when using the standard basis. This illustrates that the same location can be represented by many different coordinate systems/bases.



y
4
3
2
1
0
−1
−2
−3
−4
$\vec{x}$
−4
−3
−2
−1
0
1
2
3
4
x

4. Eigenvectors and Diagonalization

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$. Define V to be a matrix with $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

Show that $AV = V\Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

(b) **Argue that V is invertible, and therefore**

$$A = V\Lambda V^{-1}. \tag{19}$$

(Hint: Why is V invertible? It is fine to cite the appropriate result from 16A.)

(c) **Write Λ in terms of the matrices A, V, and V^{-1}.**

(d) A matrix A is deemed diagonalizable if there exists a square matrix U so that A can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix D.

Show that the columns of U must be eigenvectors of the matrix A, and that the entries of D must be eigenvalues of A.

(HINT: What does it mean to be an eigenvector? What is $U^{-1}U$? How does matrix multiplication work column-wise?)

The previous part shows that the *only* way to diagonalize A is using its eigenvalues/eigenvectors. Our method of constructing V and Λ using the eigenvalues/eigenvectors is the only way to reliably diagonalize A.

Now we will explore a payoff for diagonalizing A – an operation that diagonalization makes *much* simpler.

(e) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A \cdot A}_{k \text{ times}} \tag{20}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Conclude that A^k is diagonalizable.

5. Vector Differential Equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x} \tag{21}$$

where x_1, x_2 are scalar functions of time t, and A is a 2×2 matrix with constant coefficients. We call (21) a vector differential equation.

(a) Suppose we have a system of ordinary differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 7x_1 - 8x_2, \tag{22}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_1 - 5x_2, \tag{23}$$

Notice here that we suppress the dependence on time t for notational convenience.

Write this system in the form of (21) and compute the eigenvalues of the A matrix.

(b) **Compute the column eigenvectors of the matrix A.** For consistency, assume that the smaller eigenvalue is λ_1 and the larger is λ_2.

(c) We now want to transform our current system to a new coordinate system in order to simplify our differential equation. **What basis B should we use so that in the new coordinates $\vec{z} = B^{-1}\vec{x}$, the D matrix in the equation $\frac{\mathrm{d}\vec{z}(t)}{\mathrm{d}t} = D\vec{z}(t)$ is diagonal? Write out this new system in the $\vec{z}$ coordinates.**

(d) **Solve the new system in the $\vec{z}$ coordinates, using the initial conditions that $x_1(0) = 1, x_2(0) = -1$.**

(e) Now **convert your solution from the $\vec{z}$ coordinates back to the original $\vec{x}$ coordinates.** In other words, give us the functions $x_1(t)$ and $x_2(t)$.

(f) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (21).

Consider another second-order ordinary differential equation

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 6y(t) = 0, \tag{24}$$

First to make the problem familiar, **write the system in the form of (21), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.**

(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (24) without having to take a second derivative, and instead just taking the first derivative of something. This is another manifestation of the larger thought pattern of "lifting.")

(g) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues will have this common general form for the solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{25}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

Now **solve the system in (24) with the initial conditions** $y(0) = 1$, $\frac{\mathrm{d}y}{\mathrm{d}t}(0) = 1$**, using this method.**

(HINT: You get two equations using the initial conditions above. How many unknowns are here?) *(SECOND HINT: Given your specific choice of* x_1 *and* x_2 *in part (f), how many unknowns are there really?)*

6. Op-Amp Integrators: A continuation from the previous HW

In this question we will continue on from our analysis in Homework 2 and look at the eigenvalues of the integrator circuit (refer to Figure 5) in both non-ideal and ideal situations.

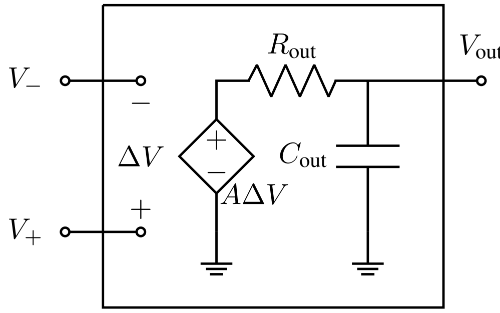
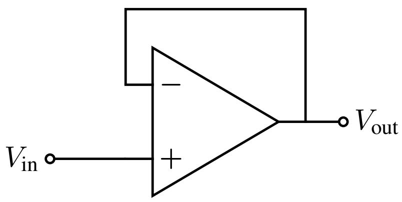
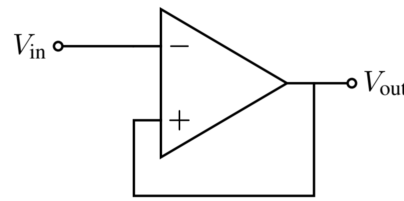


Figure 3: Op-amp model: $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback



(b) “Buffer” in positive feedback that doesn’t actually work as a buffer.

Figure 4: Op-amp in buffer configuration

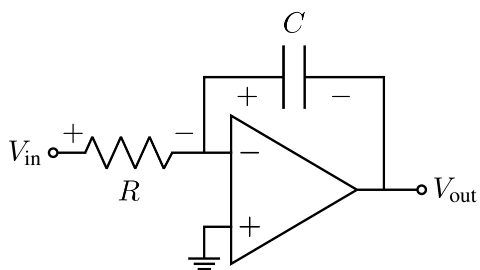


Figure 5: Integrator circuit

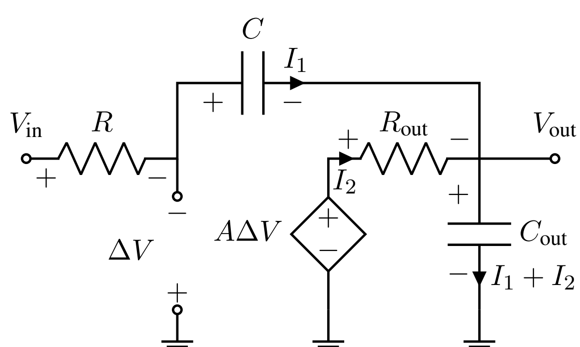


Figure 6: Integrator circuit with op-amp model.

(a) Recall from Homework 2 we had the following analysis to the integrator circuit shown in Figure 6.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_{\text{out}} \\ V_C \end{bmatrix} = \begin{bmatrix} -\left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}}\right) & -\left(\frac{1}{RC_{\text{out}}} + \frac{A}{R_{\text{out}}C_{\text{out}}}\right) \\ -\frac{1}{RC} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} V_{\text{out}} \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_{\text{out}}} \\ \frac{1}{RC} \end{bmatrix} V_{\text{in}} \tag{26}$$

Solve for the eigenvalues for the matrix/vector differential equation in Eq. (26).

For simplicity, assume $C_{\text{out}} = C = 0.01\,\text{F}$ and $R = 1\,\Omega$ and looking at the datasheet for the TI LMC6482 (the op-amps used in lab), we have $A = 10^6$ and $R_{\text{out}} = 100\,\Omega$.

Feel free to assume $A + 1 \approx 10^6$ when you finally need to plug in values at the end, but do not make any other approximations. (Of course, such an approximation is not valid if you have a $A+1-A$ term showing up somewhere.) Feel free to use a scientific calculator or Jupyter to find the eigenvalues. You don't have to grind this out by hand.

You should see that one eigenvalue corresponds to a slowly dying exponential and is close to 0. The other corresponds to a much faster dying exponential. The very slowly dying exponential is what corresponds to the desired integrator-like behavior. This is what lets it "remember." (If you don't understand why, think back to the HW problem you saw in a previous HW where you proved the uniqueness of the integral-based solution to a scalar differential equation with an input waveform.)

(b) Again, assume we have an ideal op-amp, i.e., $A \to \infty$. **Find the eigenvalues under this limit.** Feel free to make any reasonable approximations.

Here, you should see that the eigenvalue that used to be a slowly dying exponential stops dying out at all — corresponding to the ideal integrator's behavior of remembering forever. On the other hand, the eigenvalue that was very strongly negative goes straight to $-\infty$, showing the lack of interference from the other eigenvector in the ideal integrator's behavior.

7. A toy model for a solar cell

In 16A's imaging labs, you used an electronic component that responded to light in a way that could be detected electrically. To truly properly understand such things, you need to take courses like EE130 and EE134. However, in this problem, we will walk you through the modeling of a heavily simplified caricature of such a device.

In Figure 7, we illustrate what is effectively one-half of a solar cell. In simple English, what is happening is that light is striking the device and constantly causing free electron/hole pairs to be created (think of this as a kind of puddle of free charge carriers). On one side (depicted here), the electrons end up diffusing through the material until they reach a metal wire, at which point they run through the (not shown) attached circuit to meet their counterpart holes on the other side of the solar cell. The other half is symmetric, except dealing with holes. EECS 16B is a course without Physics prerequisites, and so the detailed nature of the physics here is out of scope. However, we would like to see the connection between the density of charge carriers being created by the light and the current that flows out of the solar cell.

Again, this problem is a vastly simplified caricature of what is going on in the real world, but it allows us to both get a feeling for what is happening as well as practice many core skills in 16B.

The most fundamental thing in this problem is to look at the steady state distribution of free charge carriers in the following setup, depicted in Figure 7.

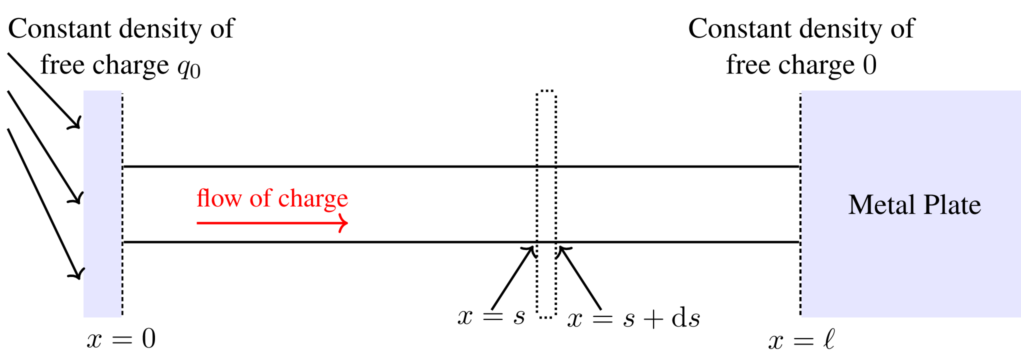


Figure 7: A hypothetical half of a toy model for a solar cell. At $x = 0$, where the light is liberating charge, the density q of free charge carriers is held constant at q_0 which depends on the intensity of the incident light. Meanwhile, at $x = \ell$, the density of free charge carriers is held constant at 0 because they get whisked away by the conducting metal wire — to race around the circuit to be reunited with their separated partners.

Let us define $q(x,t)$ as the density of charge at point x at time t.

Although we are interested in the steady-state behavior where things are going to end up not depending on t, the potential dependence on t is important to understand the differential equations that govern the behavior of the system.

In the above figure (Fig. 7), a plate with special material at $x = 0$ generates free charge carriers from light[1]. The plate is exposed to a constant light source so that the charge density $q(0,t)$ is held constant at q_0. On the rightmost end ($x = \ell$), a metal plate connected to a circuit that loops back to the other side of the solar cell forces the charge density at $x = \ell$ to be $q(\ell,t) = 0$ at all times.

Our goal is to understand what happens for the rest of $q(x,t)$ for $0 < x < \ell$.

To do so, we need to understand the dynamics that govern the behavior of charge density in the middle of this material. Physically, what is going on? What's happening is that the free charge carriers are just

[1] An example is an appropriate PN junction for a solar cell. Take 130 and/or 134 for more information!

wandering around randomly in the material. In our simplified toy model here, they have no reason to prefer moving right or left and any individual free charge carrier is just as likely to move in one direction as the other. Such random motion is called diffusion. How can we translate this into a differential equation with some predictive power?

To understand this, let's see what happens in the hypothetical small box between lengths $x = s$ and $x = s + \mathrm{d}s$ at time t. It turns out that an important quantity that we would like to understand is the *gradient* $g(x,t)$ of charge density at position x at time t,

$$g(x,t) = \frac{\mathrm{d}}{\mathrm{d}x} q(x,t). \tag{27}$$

Due to the nature of random motion, in a small time $\mathrm{d}t$, the amount of charge flowing into the box from the left $x = s$ side is equal to $-K \cdot g(s,t) \cdot \mathrm{d}t$. Here, the constant K depends on various physical constraints. You can think of it as how freely the free charge carriers are allowed to run around in the material. (Why the minus sign? Because the random flow of charge opposes the gradient of charge density — it wants to make things more level. Think of shaking a pile of sand, it will want to become less uneven by randomly flowing down the pile, not up the pile.) Meanwhile, the amount of charge entering the box from the right side $x = s + \mathrm{d}s$ is equal to $K \cdot g(s + \mathrm{d}s\,, t) \cdot \mathrm{d}t$. Hence, the change in the amount of charge that the small box gets is equal to

$$\big(-K \cdot g(s,t) \cdot \mathrm{d}t\big) + \big(K \cdot g(s + \mathrm{d}s\,, t) \cdot \mathrm{d}t\big) \approx K \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x} g(x,t) \bigg|_{x=s} \right) \cdot \mathrm{d}s \cdot \mathrm{d}t\,. \tag{28}$$

On the other hand, in a small amount of time $\mathrm{d}t$, the change in the amount of charge in the box is also $\left(\frac{\mathrm{d}}{\mathrm{d}t} q(x,t) \Big|_{x=s} \right) \cdot \mathrm{d}t \cdot \mathrm{d}s$. These must be the same, and so we can equate the two expressions to get:

$$K \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x} g(x,t) \bigg|_{x=s} \right) = \frac{\mathrm{d}}{\mathrm{d}t} q(x,t) \bigg|_{x=s}. \tag{29}$$

Since this holds for all positions s and all times t, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} q(x,t) = K \frac{\mathrm{d}}{\mathrm{d}x} g(x,t) \tag{30}$$

for some constant K that depends on the material and other physics constants.

As we can see, our knowledge of differential equations allows us to write down such a model. Equation (30) is sometimes referred to as the *heat equation* since it also models heat flow. (The role of the charge density is played by the temperature.)

In this problem, we are only interested in the steady-state case, i.e., we are going to assume that $q(x,t)$ does not change over time. That implies $\frac{\mathrm{d}}{\mathrm{d}t} q(x,t) = 0$ and using that, we can simplify our expression of $q(x,t)$ and write it as $q(x)$. Consequently, we can also simplify $g(x,t)$ to $g(x)$. Now, solving Equations (27) and (30) is equivalent to solving something we are familiar with:

$$\frac{\mathrm{d}}{\mathrm{d}x} q(x) = g(x), \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) = 0. \tag{32}$$

This is a system of differential equations of the type we know how to handle. So let's solve for both $q(x)$ and $g(x)$ from Equations (31) and (32).

(a) **Write out the differential equation in matrix/vector form:**

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} q(x) \\ g(x) \end{bmatrix} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{A} \cdot \begin{bmatrix} q(x) \\ g(x) \end{bmatrix} \tag{33}$$

Here, the "?" in the expression above simply represent the entries of the A matrix. That's what you need to fill in.

(b) **Find the eigenvalues and eigenvectors of A.**

(c) Assume that we know both $q(0)$ and $g(0)$. **Solve for $q(x)$ and $g(x)$ in terms of these initial conditions.**

(Hint: solve first for $g(x)$. What's its derivative? What does that mean?)

(d) The challenge is that the physical story does not tell us anything immediately about $g(0)$. Instead, we just know about the free carrier density at both endpoints. **Solve for $q(x)$ and $g(x)$ with boundary conditions $q(0) = q_0$ and $q(\ell) = 0$ instead.**

(Note: Solutions should be in terms of q_0, ℓ, and x.)

(Hint: If you knew $g(0)$, what would $q(\ell)$ be in terms of $q(0)$ and $g(0)$? But you know $q(\ell)$ so what does that imply?)

(e) The gradient $g(\ell)$ is related to the current flowing from the wire into the metal plate. **What is $g(\ell)$?**

(f) **Use the provided Jupyter notebook to plot your solution so that you can visualize the charge density. If you wanted to increase the current coming out of the solar cell, should you make ℓ bigger or smaller?**

(g) The above is an extremely simplified model of what happens in a solar cell. To be more realistic, you could also model the random recombination of free charge carriers within the medium itself. This recombination is proportional to the local density of free charge carriers themselves and thus modifies (30) to instead be $\frac{\mathrm{d}}{\mathrm{d}t}q(x,t) = K\frac{\mathrm{d}}{\mathrm{d}x}g(x,t) - K_2 q(x,t)$. Because we still want $\frac{\mathrm{d}}{\mathrm{d}t}q(x,t) = 0$, this changes (32) to be $\frac{\mathrm{d}}{\mathrm{d}x}g(x) = \frac{K_2}{K}q(x)$ where $K_2 > 0$ is another physical constant that depends on the material. **For this part, assume $K_2 = 1$ and $K = 16$. What is your solution for $q(x)$ in this case?**

(HINT: It is convenient here to avoid having to calculate the eigenvectors at all. From an earlier problem in this homework set, you know that the solution will have two terms to it — one corresponding to each of the distinct eigenvalues. Just get the eigenvalues and then fit to the boundary conditions.)

(h) **Use the provided Jupyter notebook to explore what happens. What does the solution tend to as $\frac{K_2}{K} \to 0$?**

8. Some more proof practice: tools for later

This problem is just some more practice with proofs, this time for results that seem obvious. We will have cause to use these results later, but for now, these will seem unmotivated as questions.

(a) Suppose we have a list of nonnegative numbers in strictly descending order $\sigma_1 > \sigma_2 > \cdots > \sigma_n \geq 0$. We want to take k of them (where $k < n$) so as to maximize their sum. **Prove that we must take the first k numbers.** (i.e. we should take the biggest numbers. Or if you want to make it practical sounding: if there are n bags of gold, and we can only carry k bags, we should pick up the k heaviest bags to maximize the gold we walk away with.)

(HINT: First, set up some notation and make precise what it is that you want to prove. Here's a path. Let α_i be a number that you can choose that is either 0 or 1 and restrict yourself to choices that satisfy $\sum_{i=1}^{n} \alpha_i = k$. Now, what you want to maximize is $\sum_{i=1}^{n} \alpha_i \sigma_i$. You want to show that the unique maximizing choice has $\alpha_i = 1$ for $i = 1, \ldots, k$ and $\alpha_i = 0$ for $i > k$. Proceed by contradiction. Suppose that you had some other selection that was claimed to be optimal and show how you could strictly improve upon it. If you could improve upon it, it couldn't have been optimal.)

(b) Now we relax our choice. Suppose that $\alpha_1, \ldots, \alpha_n$ are real numbers, with $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{n} \alpha_i \leq k$. We want to maximize $\sum_{i=1}^{n} \alpha_i \sigma_i$. **Show that the unique maximizing choice has $\alpha_i = 1$ for $i = 1, \ldots, k$ and $\alpha_i = 0$ for $i > k$.**

(HINT: Proceed with the same style of argument as above. Suppose that you had some other values β_i that satisfied the constraints above and was claimed to be optimal and show how you could strictly improve upon it. If you could improve upon it, it couldn't have been optimal.)

(NOTE: The previous part is a version of this part where α_i is either 0 or 1. In this way α_i tells us how much of σ_i we choose to take.)

9. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

10. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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