EECS 16B Designing Information Devices and Systems II
Fall 2021 UC Berkeley

Homework 04

This homework is due on Friday, September 24, 2021, at 11:59PM. Self-grades and HW Resubmission are due on Tuesday, September 28, 2021, at 11:59PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note -1, Note 4, and Note 5.

   (a) **What is the equation that governs the relationship that an inductor enforces on the voltage across it and the current through it? What is the behavior of an inductor under DC current (i.e. constant current)?**

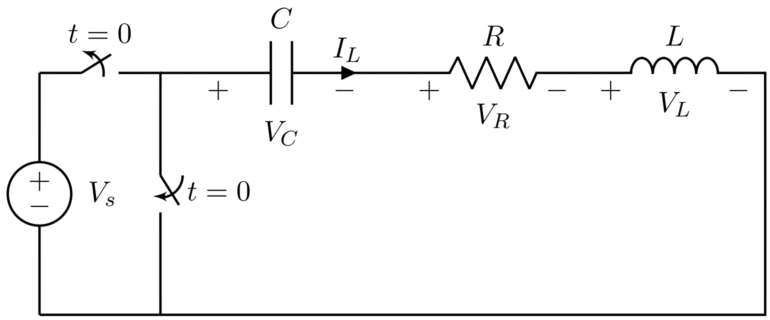
   (b) **What is the definition of impedance? What quantity is impedance analogous to in the static DC circuit analysis you learned in 16A?**

   (c) On Monday, 16B released some official guidance and recommendations regarding study tips and strategies to approach this class in Note -1.

   Please comment on how you intend to follow the guidance or whether you intend to do something different, with the full understanding that by doing so, you are putting your performance in the class at risk.

2. RLC Responses: Initial Part

Consider the following circuit:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

The sequence of problems 2 - 6 combined will try to show you the various RLC system responses and how they relate to how the eigenvalues of the A matrix changes. Note that the work you will do will also hold for any second-order system, like a mass-spring-damper, and is very common to study in controls as we'll see later on in Module 2 of the course.

(a) We first need to construct our state space system. Our natural state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the quantities whose derivatives show up in the system of equations governing our circuit. Now, **find the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$.** Leave the system symbolic in terms of V_s, L, R, and C.

(b) **Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$.** This should be in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

(c) **Find the eigenvalues of the A matrix symbolically.**
(Hint: the quadratic formula will be involved.)

(d) **Under what condition on the circuit parameters R, L, C are there going to be a pair of distinct purely real eigenvalues of A?**

(e) **Under what condition on the circuit parameters R, L, C are there going to be a pair of purely imaginary eigenvalues of A? What will the eigenvalues be in this case?**

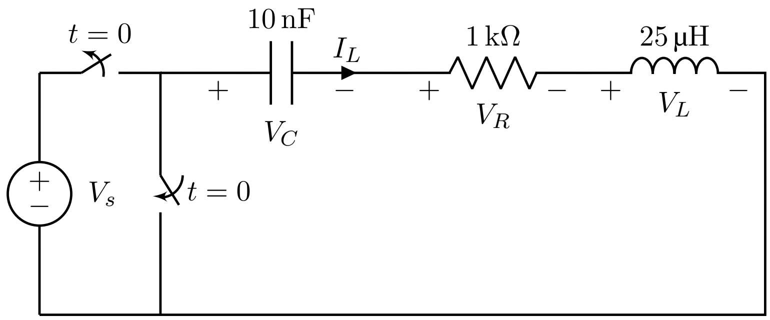
(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex when the conditions of part (d) fail to hold) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, **find eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ corresponding to them.** The final answer can be in terms of λ_1, λ_2 and other constants.

(HINT: Rather than trying to find the relevant nullspaces, etc., try to find eigenvectors of the form $\begin{bmatrix} 1 \\ y \end{bmatrix}$ where we just want to find the missing entry y. This works because we know the first entry of the eigenvector can not be 0 and we want to normalize it to so that first entry is 1.)

(g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **What is the $\widetilde{A}$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$?** It is fine to have your answer expressed symbolically using λ_1, λ_2.

3. RLC Responses: Overdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

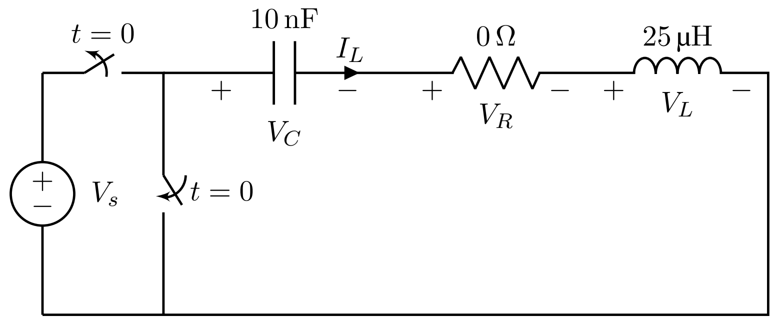
(a) Suppose $R = 1\,\text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.**

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 1\,\text{k}\Omega$ and $C = 10\,\text{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane.**

4. RLC Responses: Undamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume that the capacitor is charged to V_s and there is no current in the inductor for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Suppose $R = 0\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.** Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

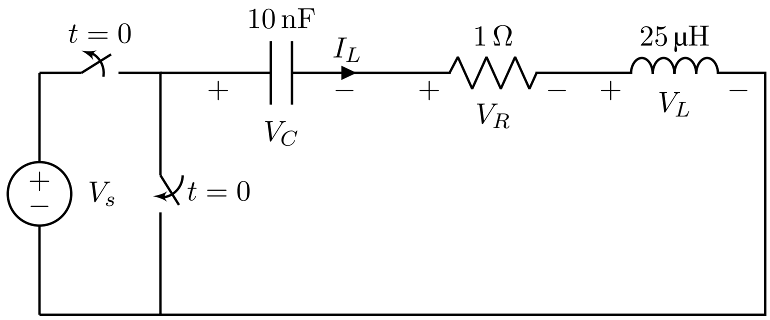
(HINT: Use Euler's formula.)

(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 0\,\Omega$ and $C = 10\,\text{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?**

Note: Because there is no resistance, this is called the "undamped" case.

5. RLC Responses: Underdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may round numbers to make the algebra more simple. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) Now suppose that $R = 1\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1\,\text{V}$. **Find the initial conditions for $\vec{\tilde{x}}(0)$.** Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

(b) Using the diagonalized system from 2(g) and continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.** Remember that your final expressions for $x_1(t)$ and $x_2(t)$ should be real functions (no imaginary terms).

(HINT: Remember that $e^{a+jb} = e^a e^{jb}$. Use Euler's formula.)

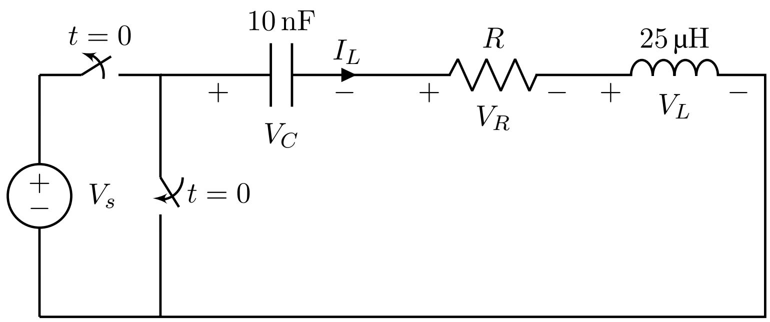
(c) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. Do the waveforms for $x_1(t)$ and $x_2(t)$ decay to 0?**

Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

(d) Notice that you got answers in terms of complex exponentials. **Why did the final voltage and current waveforms end up being purely real?**

6. RLC Responses: Critically Damped Case

Building on the previous problem, consider the following circuit with specified component values: (Notice R is not specified yet. You'll have to figure out what that is.)



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may use a calculator or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

(a) **For what value of R is there going to be a single eigenvalue of A?**

(b) Using the given values for the capacitor and the inductor, as well as the critical value for the resistance R that you found in the previous part, **find the eigenvalues and eigenspaces of A. What are the dimensions of the corresponding eigenspaces?** (i.e. how many linearly independent eigenvectors can you find associated with this eigenvalue?)

It is easier to do the algebra with a non-symbolic matrix to work with.

(c) We now create a new coordinate system V, with the first vector being $\vec{v}_\lambda$ — the eigenvector you found for the single eigenvalue λ above. For the second vector, just pick $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for lack of a principled choice. (We will see later why this choice is ok.) We then apply a change of basis to define $\vec{\widetilde{x}}$ in the transformed coordinates such that $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. **What is the resulting $\widetilde{A}$ matrix defining the system of differential equations in the transformed coordinates?**

(d) Notice that the second differential equation for $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_2(t)$ in the above coordinate system only depends on $\widetilde{x}_2(t)$ itself. There is no cross-term dependence. This happened because we earlier chose $\vec{v}_1$ such that the transformation $\widetilde{A}$ becomes upper triangular and this results in the removal of any cross-dependency. (We will later see that in fact, we had many other choices for $\vec{v}_2$ — as far as $\vec{v}_2$ was concerned, we aren't looking for a needle in a haystack, we're looking for hay in a haystack.) Now, **compute the initial condition for $\widetilde{x}_2(0)$ and write out the solution to this scalar differential equation for $\widetilde{x}_2(t)$ for $t \geq 0$.** Assume that $V_s = 1\,\mathrm{V}$.

(e) With an explicit solution to $\widetilde{x}_2(t)$ in hand, substitute this in, **write out the resulting scalar differential equation for $\widetilde{x}_1(t)$.** This should effectively have a time-dependent input in it. **Also compute the initial condition for $\widetilde{x}_1(0)$.**

Note: this is just the differential-equations counterpart to the back-substitution step from Gaussian Elimination in 16A. Once you had done one full downward pass of Gaussian Elimination, you went upwards and just substituted in the solution that you found to remove this dependence from the equations above. This is the exact same design pattern, except for a system of linear differential equations.

(f) **Solve the above scalar differential equation with input and write out what $\widetilde{x}_1(t)$ is for $t \geq 0$.**
(HINT: You might want to look at a problem on an earlier homework for help with this.)

(g) **Find $x_1(t)$ and $x_2(t)$ for $t \geq 0$ based on the answers to the previous three parts.**
This particular case is called the "critically damped case" for an RLC circuit. It is called this because the R value you found demarcates the boundary between solutions of the underdamped and overdamped variety.

(h) In the `RLCSliders.ipynb` Jupyter notebook, move the sliders to the resistance value you found in the first part and $C = 10\,\mathrm{nF}$. **Comment on the graph of $V_C(t)$ and the location of the eigenvalues on the complex plane. What happens to the voltage and eigenvalues as you slightly increase or decrease R?**

(i) In part (c) we saw that A only had one eigenvalue, λ, and one eigenvector, $\vec{v}_\lambda$. This meant that we had a choice for $\vec{v}_2$ in the expression $V = \begin{bmatrix} \vec{v}_\lambda & \vec{v}_2 \end{bmatrix}$. We seemingly arbitrarily chose $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We claim that there are many correct choices of $\vec{v}_2$ that will result in $\widetilde{A} = V^{-1}AV = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ (i.e. it is upper triangular). Remember, we want $\widetilde{A}$ to be upper triangular so that we have an uncoupled differential equation for $\vec{\widetilde{x}}_2(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{bmatrix} \implies \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_2(t) = c\widetilde{x}_2(t) \tag{1}$$

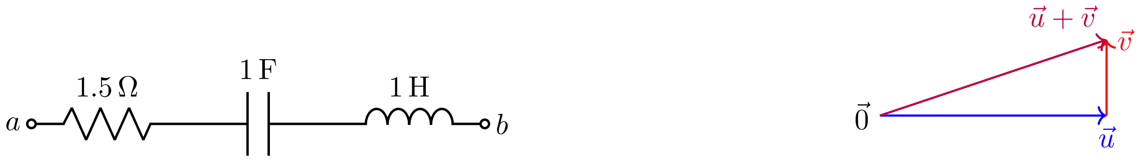
In fact, it turns out that we can pick any $\vec{v}_2$ as long as $\vec{v}_2 \neq k\vec{v}_\lambda$ for some $k \in \mathbb{R}$. We will try and prove this very claim. More concisely, prove the statement below:

if $V = \begin{bmatrix} \vec{v}_\lambda & \vec{v}_2 \end{bmatrix}$ and $\vec{v}_2 \neq k\vec{v}_\lambda$ for some $k \in \mathbb{R}$, then $\widetilde{A} = V^{-1}AV = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

(HINT: The key is to establish the first column — namely $\begin{bmatrix} a \\ 0 \end{bmatrix}$. What does that column mean? What does the first column in a matrix mean? Surely you are going to have to use the only special thing that you know — namely that $\vec{v}_\lambda$ is an eigenvector.)

7. Phasors

The impedances for a resistor, R, inductor, L, and capacitor, C, are $Z_R(\mathrm{j}\omega) = R$, $Z_L(\mathrm{j}\omega) = \mathrm{j}\omega L$, and $Z_C(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega C} = -\frac{\mathrm{j}}{\omega C}$ respectively. Consider the following components connected in series between terminals a and b in figure 1a: a resistor, capacitor, and inductor.

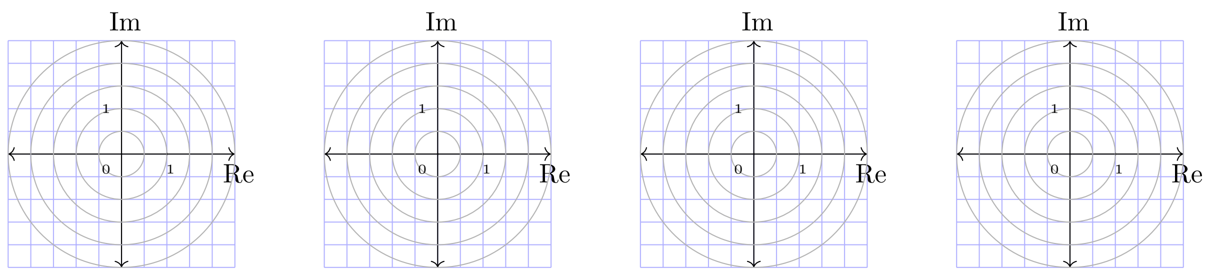


(a) Three components in series.

(b) Vector sum of $\vec{u}$ and $\vec{v}$.

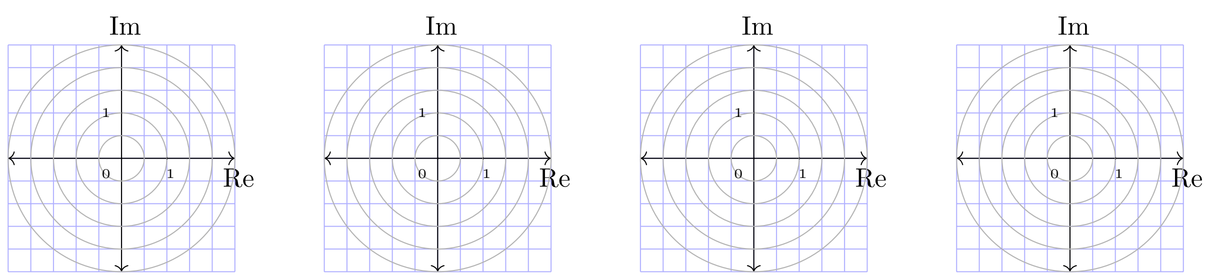
Figure 1: Relevant problem figures.

(a) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = \frac{1}{2}\,\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



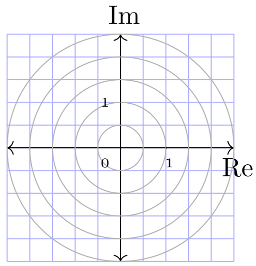
(a) $Z_R(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 0.5\,\frac{\text{rad}}{\text{s}})$

(b) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = 1\,\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**

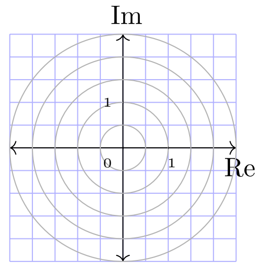


(a) $Z_R(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(b)** $Z_C(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(c)** $Z_L(@\omega = 1\,\frac{\text{rad}}{\text{s}})$ **(d)** $Z_{\text{ab}}(@\omega = 1\,\frac{\text{rad}}{\text{s}})$

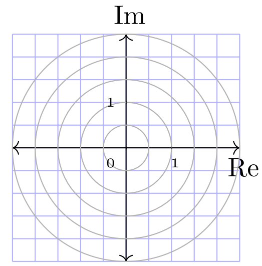
(c) **For the component values given in figure 1a, evaluate the impedances Z_R, Z_C, Z_L and the series equivalent impedance Z_{ab} for the case $\omega = 2\,\frac{\text{rad}}{\text{s}}$. Draw the individual impedances as "vectors" on the complex plane. On the last plot draw Z_{ab} as a vector sum (as shown in figure 1b) of Z_R, Z_C, and Z_L on the complex plane. Then give the magnitude and phase of Z_{ab}.**



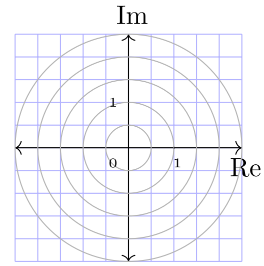
(a) $Z_R(@\omega = 2\ \frac{\text{rad}}{\text{s}})$



(b) $Z_C(@\omega = 2\ \frac{\text{rad}}{\text{s}})$



(c) $Z_L(@\omega = 2\ \frac{\text{rad}}{\text{s}})$



(d) $Z_{\text{ab}}(@\omega = 2\ \frac{\text{rad}}{\text{s}})$

(d) The "natural frequency" ω_n is defined as the frequency ω_n where the net impendance is purely real. **For the series combination of RLC elements, Z_{ab}, appearing in figure 1a, what is the "natural frequency" ω_n?**

Fact: We call this the "natural frequency" since it is the frequency at which the magnitude of the impedance is the smallest. It turns out to be the case that such a circuit will oscillate at this frequency if it was underdamped (if R was small enough) and we set it up in a problem like that of the underdamped problem on this HW set.

8. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning (from the bottom up) is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

You need to write your own question and provide a thorough solution to it. The scope of your question should roughly overlap with the scope of this entire problem set. This is because we want you to exercise your understanding of this material, and not earlier material in the course. However, feel free to combine material here with earlier material, and clearly, you don't have to engage with everything all at once. A problem that just hits one aspect is also fine.

Note: One of the easiest ways to make your own problem is to modify an existing one. Ordinarily, we do not ask you to cite official course materials themselves as you solve problems. This is an exception. Because the problem making process involves creative inputs, you should be citing those here. It is a part of professionalism to give appropriate attribution.

Just FYI: Another easy way to make your own question is to create a Jupyter part for a problem that had no Jupyter part given, or to add additional Jupyter parts to an existing problem with Jupyter parts. This often helps you learn, especially in case you have a programming bent.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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