EECS 16B Designing Information Devices and Systems II
Fall 2021 UC Berkeley

Homework 05

This homework is due on Friday, October 1, 2021, at 11:59PM. Self-grades and HW Resubmission are due on Tuesday, October 5, 2021, at 11:59PM.

1. **Reading Lecture Notes**

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 5 and Note 6.

(a) Consider an RC circuit with a sinusoidal voltage input $V(t) = A\cos(\omega t)$. We are interested in finding the voltage on the capacitor in steady state (after a long time has passed). **Can we solve this using our standard differential equation techniques? Can we solve this with phasors? Which one is more concise and why?**

(b) There are two ways to make a low pass filter (discussed in the notes). **What are they?**

(c) **Draw the voltage sources between terminals a and b in figure 1 as a single equivalent voltage source between terminals a and b, and label its voltage value. How does this equivalence relate to filtering and phasor analysis?**

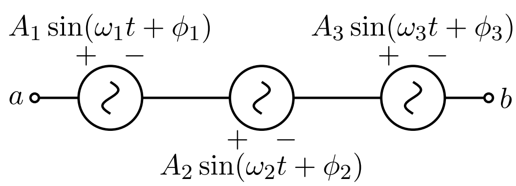


Figure 1: Three voltage sources in series.

(d) **How we can address filter loading?**

2. Group Re-assignment Survey

How are your study groups working out?

(If you don't have a study group you can just say so for full credit.)

We hope they have been helpful so far. If you feel things are not going as well as you hoped and you would prefer to be assigned to a new group, or if you did not request a group before but have decided you would like one going forward, please fill out the following form:

(a) Group Re-assignment Survey - Google Form

3. Low-pass Filter

You have a $1\,\mathrm{k}\Omega$ resistor and a $1\,\mu\mathrm{F}$ capacitor wired up as a low-pass filter.

(a) **Draw the filter circuit, labeling the input node, output node, and ground.**

(b) **Write down the transfer function of the filter, $H(\mathrm{j}\omega)$ that relates the output voltage phasor to the input voltage phasor.** Be sure to use the given values for the components.

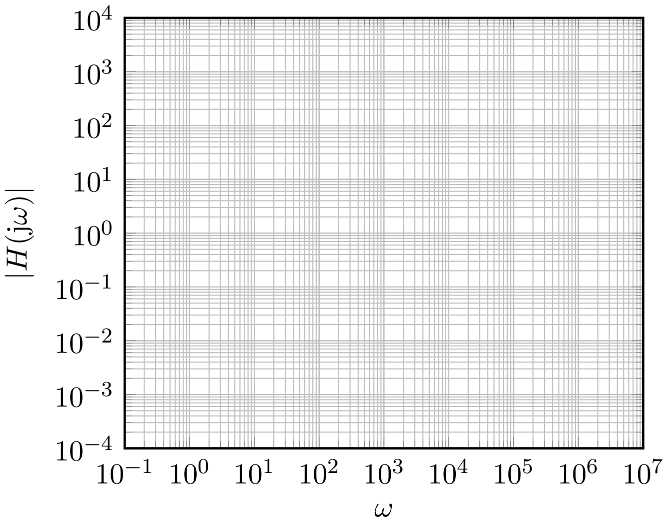
(c) **Write an exact expression for the *magnitude* of $H(\mathrm{j}\omega = \mathrm{j}10^6)$, and give an approximate numerical answer.**

(d) **Write an exact expression for the *phase* of $H(\mathrm{j}\omega = \mathrm{j}1)$, and give an approximate numerical answer.**

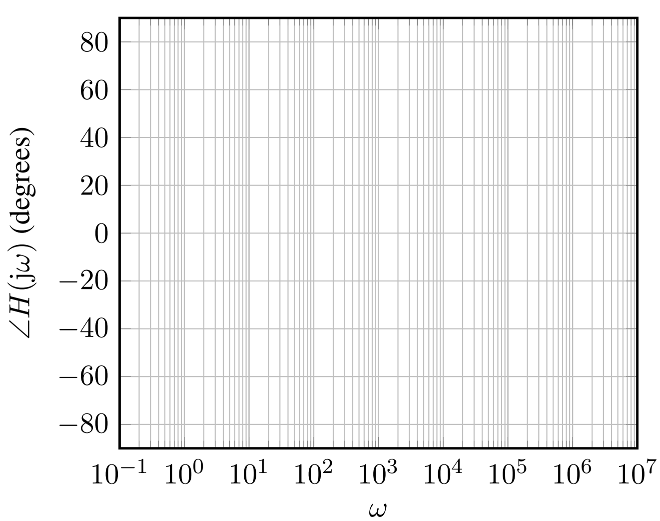
(e) **Write down an expression for the time-domain output waveform $V_{\mathrm{out}}(t)$ of this filter if the input voltage is $V(t) = 1\sin(1000t)$ V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

(f) **Use a computer or calculator to help you sketch the transfer function (both magnitude and phase) of the filter on the graph paper below.**

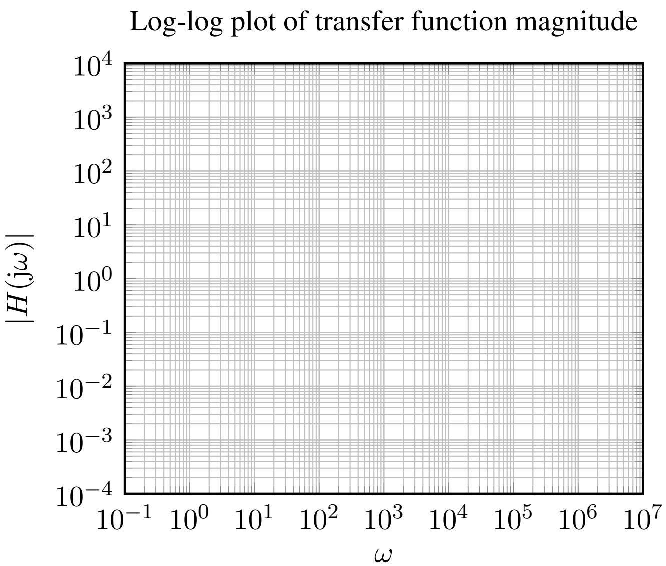
Log-log plot of transfer function magnitude

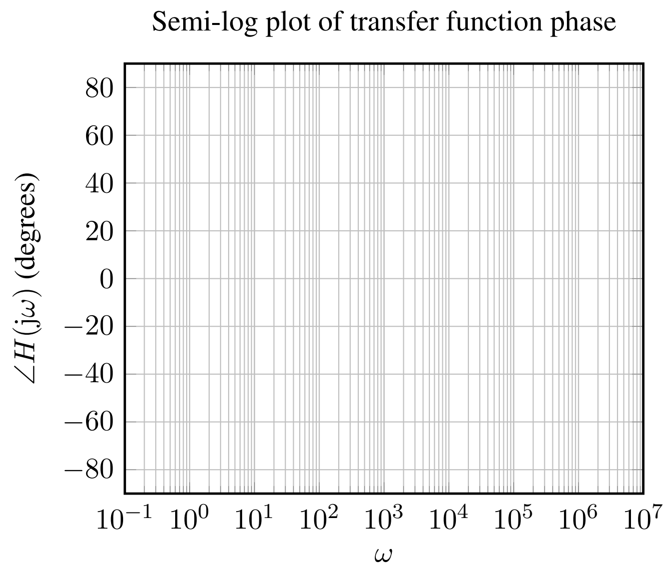


Semi-log plot of transfer function phase



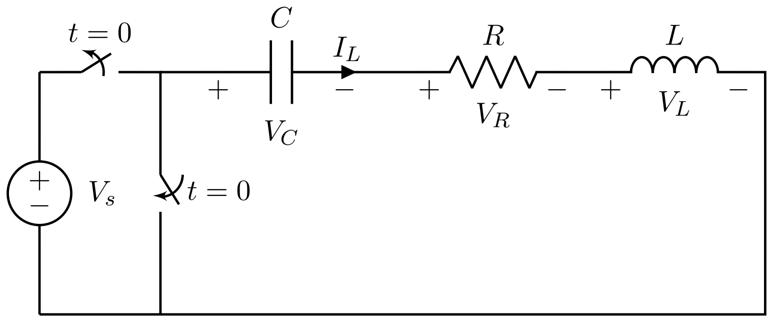
(g) Now, let's consider that the $1\,\mathrm{k}\Omega$ resistor and the $1\,\mu\mathrm{F}$ capacitor are wired up as a high-pass filter. **Draw the high-pass filter circuit with labels. Use a computer or calculator to help you sketch the transfer function (both magnitude and phase) of the high-pass filter on the graph paper below.**


Log-log plot of transfer function magnitude
$|H(j\omega)|$
10^{4} 10^{3} 10^{2} 10^{1} 10^{0} 10^{-1} 10^{-2} 10^{-3} 10^{-4}
10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5} 10^{6} 10^{7}
ω


Semi-log plot of transfer function phase
$\angle H(j\omega)$ (degrees)
80 60 40 20 0 −20 −40 −60 −80
10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5} 10^{6} 10^{7}
ω

4. Alternative "second order" perspective on solving the RLC circuit

Consider the following circuit like you saw in lecture, discussion, and the previous homework:



Suppose now we insisted on expressing everything in terms of one waveform $V_C(t)$ instead of two of them (voltage across the capacitor and current through the inductor). This is called the "second-order" point of view, for reasons that will soon become clear.

For this problem, use R for the resistor, L for the inductor, and C for the capacitor in all the expressions until the last part.

(a) **Write the current $I_L(t)$ through the inductor in terms of the voltage $V_C(t)$ across the capacitor.**

(b) Now, notice that the voltage drop across the inductor involves $\frac{\mathrm{d}}{\mathrm{d}t} I_L(t)$. **Write the voltage drop across the inductor, $V_L(t)$, in terms of the second derivative of $V_C(t)$.**

(c) For this part, treat $V_s(t)$ as a generic input waveform. Consider the switch to be in the same configuration for all t, corresponding to $t < 0$ for the previous parts.

Now write out a differential equation governing $V_C(t)$ in the form of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} V_C(t) + a\frac{\mathrm{d}}{\mathrm{d}t} V_C(t) + bV_C(t) + c(t) = 0. \tag{1}$$

where a, b and $c(t)$ are terms you need to figure out by analyzing the circuit.

(HINT: The $c(t)$ needs to involve $V_s(t)$ in some way.)

(d) If we hadn't done earlier homework, we wouldn't know how to solve equations like eq. (1). But to reduce this to something we know how to solve, we define $X(t)$ as an additional state, with $X(t) = \frac{\mathrm{d}}{\mathrm{d}t} V_C(t)$. Note that this definition directly gives us one equation: $\frac{\mathrm{d}}{\mathrm{d}t} V_C(t) = X(t)$. This leaves us needing an equation for $\frac{\mathrm{d}}{\mathrm{d}t} X(t)$. **Express $\frac{\mathrm{d}}{\mathrm{d}t} X(t)$ in terms of $X(t)$, $V_C(t)$, and $V_s(t)$. Write a matrix differential equation in terms of $V_C(t)$ and $X(t)$.** Your answer should be in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{A} \begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} V_s(t). \tag{2}$$

(e) **What is the characteristic polynomial of the matrix A from eq. (2)? Comment on any relationship it might have to the second-order differential equation eq. (1) you found earlier.**

(f) **Find the eigenvalues (and OPTIONALLY eigenvectors) of the matrix A from eq. (2).**

(Hint: use the same trick you did in the previous homework, i.e., look for eigenvectors of the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$.

Please don't look for eigenvectors the hard way.)

(g) **Revisit Problem 3 (and OPTIONALLY Problem 5) of the previous homework, and use the values of R, L, C, and the same initial conditions to solve for V_C.** Did you get the same answer as in Problem 3 (and optionally Problem 5) of the previous homework?

(For fun: Remember Problem 5 part (g) of HW 3. Can you use those ideas here to solve the problem without having to use the eigenvectors explicitly?)

5. Color Organ Filter Design

The "color organ" is a three-class tone classifier that we will hand design in the fourth lab. We will design low-pass, band-pass, and high-pass filters for our color organ. There will be red, green, and blue LEDs. Each color will correspond to a specified frequency range of the input audio signal. The intensity of the light emitted will correspond to the amplitude of the audio signal.

(a) First, you remember that you saw in lecture that you can build simple filters using a resistor and a capacitor. **Design a simple first-order *passive* low-pass filter with the following specification using a $1\,\mu\text{F}$ capacitor.** ("Passive" means that the filter does not require any power supply to operate on the input signal. Passive components include resistors, capacitors, inductors, diodes, etc., while an example of an active component would be an op-amp).

- Low-pass filter: cut-off frequency $f_c = 2400\,\text{Hz}$, $\omega_c = 2\pi \cdot 2400\,\frac{\text{rad}}{\text{s}}$

Recall that the cutoff-frequency of such a filter is just where the magnitude of the filter is $\frac{1}{\sqrt{2}}$ of its peak value.

Show your work to find the resistor value that creates this low-pass filter. Draw the schematic-level representation of your design. Please mark V_{in}, V_{out}, and the ground node(s) in your schematic. Round your results to two significant figures.

(b) **Now design a simple first-order *passive* high-pass filters with the following specification using a $1\,\mu\text{F}$ capacitor.**

- High-pass filter: cut-off frequency $f_c = 100\,\text{Hz}$, $\omega_c = 2\pi \cdot 100\,\frac{\text{rad}}{\text{s}}$

Show your work to find the resistor value that creates this high-pass filter. Draw the schematic-level representation of your design. Please mark V_{in}, V_{out}, and the ground node(s) in your schematic. Round your results to two significant figures.

(c) You can try to build a bandpass filter by cascading the first-order low-pass and high-pass filters you designed in parts (a) and (b). To do this, you might be tempted to connect the V_{out} node of your low-pass filter directly to the V_{in} node of your high-pass filter. However, if you did this, just as you saw in 16A for voltage dividers, the purported high-pass filter would "load" the low-pass filter and you might get some potentially complicated mess instead of what you wanted.

Show how you can use an ideal op-amp configured as a unity gain buffer to eliminate this loading effect to cascade the low-pass and high-pass filters, and write the resulting transfer function of the combined circuit. Draw the magnitude and phase transfer functions of the combined circuit. What kind of filter is this? You can optionally use the included Jupyter notebook `plot_tf.ipynb`. *(HINT: Read Section 2.1 in Note 7.)*

(NOTE: In Python, use 1j *when your transfer function has a* j.*)*

(d) **Write down an expression for the time-domain output waveform $V_{\text{out}}(t)$ of this filter if the input voltage is $V_{\text{in}}(t) = 1\sin(1000t)\text{V}$.** Round your answer to 2 significant digits.

6. Phasors and Eigenvalues

Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{3}$$

where for this problem, the matrix A and the vector $\vec{b}$ are both real.

(a) **Give a necessary condition on the eigenvalues λ_k of A such that any impact of an initial condition will eventually completely die out.** (i.e. the system will reach steady-state.)

You don't have to prove this. The idea here is to make sure that you understand what kind of thing is required. *(HINT: Read Section 2 in Note 5.)*

(b) Now assume that $u(t)$ has a phasor representation $\widetilde{U}$. In other words, $u(t) = \widetilde{U}\mathrm{e}^{+\mathrm{j}\omega t} + \overline{\widetilde{U}}\mathrm{e}^{-\mathrm{j}\omega t}$.

Assume that the vector solution $\vec{x}(t)$ to the system of differential equations (3) can also be written in phasor form as

$$\vec{x}(t) = \widetilde{\vec{X}}\mathrm{e}^{+\mathrm{j}\omega t} + \overline{\widetilde{\vec{X}}}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{4}$$

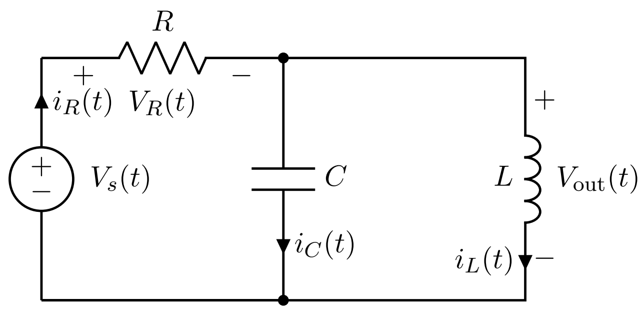
Derive an expression for $\widetilde{\vec{X}}$ involving $A, \vec{b}$, $\mathrm{j}\omega$, $\widetilde{U}$, and the identity matrix I.

(HINT: Plug (4) into (3) and simplify, using the rules of differentiation and grouping terms by which exponential $\mathrm{e}^{\pm \mathrm{j}\omega t}$ they multiply.)

7. Phasor-Domain Circuit Analysis

The analysis techniques you learned previously in 16A for resistive circuits are equally applicable for analyzing circuits driven by sinusoidal inputs in the phasor domain. In this problem, we will walk you through the steps with a concrete example.

Consider the following circuit where the input voltage is sinusoidal. The end goal of our analysis is to find an equation for $V_{\text{out}}(t)$.



The components in this circuit are given by:

$$V_s(t) = 10\sqrt{2}\cos\left(100t - \frac{\pi}{4}\right) \tag{5}$$

$$R = 5\,\Omega \tag{6}$$

$$L = 50\,\text{mH} \tag{7}$$

$$C = 2\,\text{mF} \tag{8}$$

(a) **Give the amplitude V_0, input frequency ω, and phase ϕ of the input voltage V_s.**

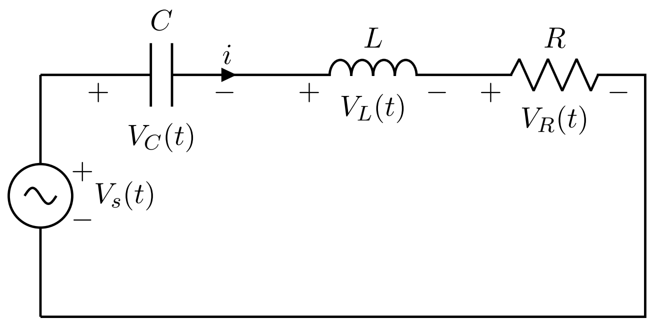
(b) Transform the circuit into the phasor domain. **What are the impedances of the resistor, capacitor, and inductor? What is the phasor $\widetilde{V}_S$ of the input voltage $V_s(t)$?**

(c) Use the circuit equations to **solve for** $\widetilde{V}_{\text{out}}$, the phasor representing the output voltage.

(d) **Convert the phasor $\widetilde{V}_{\text{out}}$ back to get the time-domain signal $V_{\text{out}}(t)$.**

8. RLC filter

Consider the following RLC circuit:



(a) **Write down the impedance of a series RLC circuit in the form $Z_{RLC}(j\omega) = A(\omega) + \mathrm{j}X(\omega)$, where $X(\omega)$ is a real valued function of ω.**

(b) **Write the transfer function from V_S to V_R — the voltage drop across the resistor.**

(c) **For the different specific values for R, L, C given by different cases (underdampled, overdamped, and critically damped) in the previous HW, use a computer to sketch plots of the magnitude and phase of the transfer function above.** You can optionally use the included Jupyter notebook `plot_tf.ipynb`. *(NOTE: In Python, use* 1j *when your transfer function has a* j.*)*

(d) To see how the values of R, L, C impact the impedance at different frequencies, run the included Jupyter notebook **hw5rlc_ transfer.ipynb**. The script will generate two plots, the transfer function of the circuit as a function of frequency and the location of the eigenvalues in the imaginary, real plane. **Explain what happens at the following sets of values as the resistance, inductance, and capacitance vary:**

	R	L	C
I	1	2.5E−5	1E−8
II	10	2.5E−5	1E−8
III	10	2.5E−5	2E−9
IV	500	0.0001	2E−8

Table 1: Values for RLC Bandwidth problem, part d

9. Uniqueness justification for phasor-style solutions

In general, we have seen that we need to justify our methods of solving differential equations with a uniqueness proof. This important so that it tells us we can trust our solution as being the only solution to the problem at hand.

In Note 5, you saw that phasor-style techniques gave rise to convenient solutions to the matrix-vector differential equations of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \tilde{\vec{U}}\mathrm{e}^{st} \tag{9}$$

with some initial condition $\vec{x}(0) = \vec{x}_0$.

However, all the uniqueness proofs that you have done for yourself have been concerned with scalar differential equations, and scalar differential equations driven by inputs. So, why can we trust the solutions that we are getting for such matrix-vector differential equations?

This question takes us part of the way to the answer.

(a) Suppose that the $n \times n$ matrix A has n distinct eigenvalues and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, so that the matrix $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}$ has linearly independent columns.

Show that if you are given any valid solution for the original system (9), you can change coordinates to the eigenbasis and also get a valid solution for the diagonalized system. Additionally, list the new initial conditions that are satisifed in the diagonalized system.

(b) You have already proved the uniqueness of solutions for any scalar differential equation of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$. How can you use this fact and the result of the previous part to argue that the solution must be unique for the matrix/vector differential equation?

Give a proof by contradiction argument.

(Hint: Start by assuming that you have two distinct solutions. Use the linear independence of V and the fact that a change of basis is invertible.)

We will see later in the course how the assumption we made on the eigenvectors of A is not actually needed for this proof to hold. But for now, it is important to understand this case first.

10. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning (from the bottom up) is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

You need to write your own question and provide a thorough solution to it. The scope of your question should roughly overlap with the scope of this entire problem set. This is because we want you to exercise your understanding of this material, and not earlier material in the course. However, feel free to combine material here with earlier material, and clearly, you don't have to engage with everything all at once. A problem that just hits one aspect is also fine.

Note: One of the easiest ways to make your own problem is to modify an existing one. Ordinarily, we do not ask you to cite official course materials themselves as you solve problems. This is an exception. Because the problem making process involves creative inputs, you should be citing those here. It is a part of professionalism to give appropriate attribution.

Just FYI: Another easy way to make your own question is to create a Jupyter part for a problem that had no Jupyter part given, or to add additional Jupyter parts to an existing problem with Jupyter parts. This often helps you learn, especially in case you have a programming bent.

11. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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