EECS 16B   Designing Information Devices and Systems II

Fall 2021   UC Berkeley   Homework 07

This homework is due on Friday, October 15, 2021, at 11:59PM. Self-grades and HW Resubmission are due on Tuesday, October 19, 2021, at 11:59PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 10 and Note 11.

   (a) **How would you use feedback control to choose the closed-loop eigenvalues of a closed-loop discrete-time system?**

   (b) **What is the matrix test for controllability of a general linear discrete-time system $\vec{x}[i + 1] = A\vec{x}[i] + \vec{b}u[i]$ with a scalar input $u[i]$?**

   (c) **If $\vec{b}$ above were an eigenvector of A, why would this imply that the system is not controllable if the dimension of $\vec{x}$ is larger than 1?**

2. Stability Criterion

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.
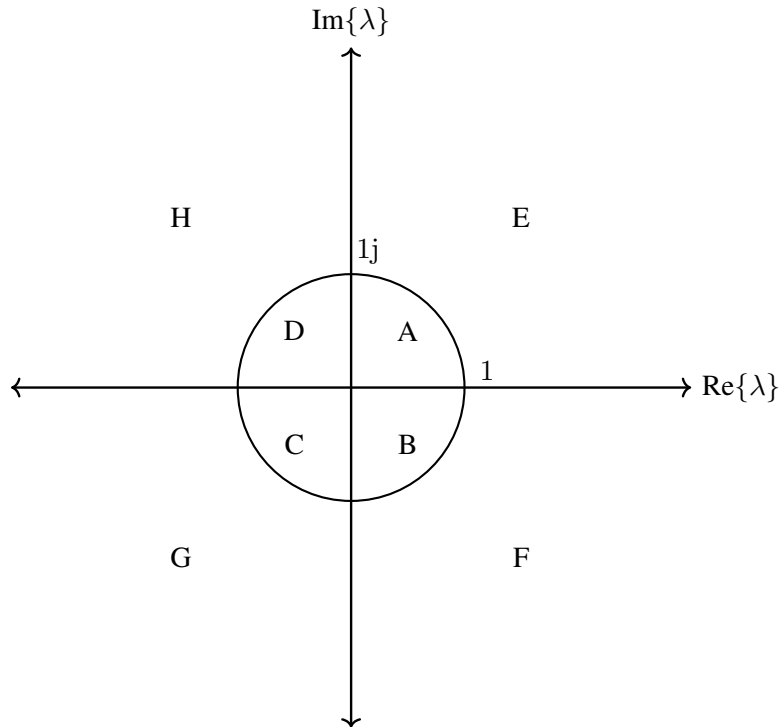

Figure 1: Complex plane divided into regions.

(a) Consider the continuous-time system $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + v(t)$
and the discrete-time system $y[i+1] = \lambda y[i] + w[i]$. Here $v(t)$ and $w[i]$ are both disturbances to their respective systems.

In which regions can the eigenvalue λ be for the system to be *stable*? Fill out the table below to indicate *stable* regions. Assume that the eigenvalue λ does not fall directly on the boundary between two regions.

	A	B	C	D	E	F	G	H
Continuous Time System $x(t)$	○	○	○	○	○	○	○	○
Discrete Time System $y[i]$	○	○	○	○	○	○	○	○

3. BIBO Stability

(a) Consider the circuit below with $R = 1\Omega$, $C = 0.5\text{F}$.
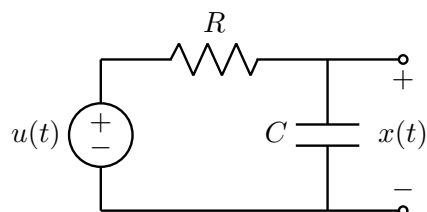
We know the circuit can be modeled by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -2x(t) + 2u(t) \tag{1}$$

Show that this system is BIBO stable meaning $x(t)$ remains bounded for all time if the input $u(t)$ is bounded. Equivalently, assume $|u(t)| < k$, $\forall t \geq 0$ and show $|x(t)| < M$, $\forall t \geq 0$ for some constant M. Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

HINT: You may want to write the expression for $x(t)$ in terms of $u(t)$ and $x(0)$ and then take the norms of both sides to show a bound on $|x(t)|$. Remember that norm in 1D is absolute value. Some helpful formulas are $|ab| = |a||b|$, the triangle inequality $|a + b| \leq |a| + |b|$, and the integral version of the triangle inequality $\left|\int_a^b f(\tau)\,\mathrm{d}\tau\right| \leq \int_a^b |f(\tau)|\,\mathrm{d}\tau$, which just extends the standard triangle inequality to an infinite sum of terms.

(b) Consider a continuous-time scalar real differential equation with known solution

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t) \qquad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau. \tag{2}$$

Show that if the system has $\mathrm{Re}\{a\} > 0$, then the system is BIBO unstable, so there exists a bounded input $\left(|u(t)| \leq \epsilon\right)$ that can result in an unbounded output. Assume $x(0) = 0$.

(c) **Repeat the previous part to show the system is BIBO unstable for the specific case of purely imaginary $a = j2\pi$. Again assume $x(0) = 0$.**

All the other purely imaginary unstable cases are the same way for the same essential reason.

(d) We now consider the discrete-time system

$$x[i + 1] = -3x[i] + u[i] \tag{3}$$

with $x[0] = 0$.

Is this system stable or unstable? If stable, prove it. If unstable, find a bounded input sequence $u[i]$ that causes the system to grow unbounded.

(e) For the example in the previous part, **give an explicit sequence of inputs that are not zero but for which the state $x[i]$ will always stay bounded.** *(HINT: See if you can find any input pattern that results in an oscillatory behavior.)*

(f) Consider the discrete-time real system with known solution:

$$x[i + 1] = ax[i] + bu[i] \qquad x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^{i-1-k} bu[k] \tag{4}$$

Show that if $|a| > 1$, then a bounded input can result in an unbounded output, i.e. the system is BIBO unstable. Assume that $x[0] = 0$.

(g) **[Optional]** since this part was derived in lecture. Now consider the discrete-time stable case where a is complex and has $|a| < 1$. **Show that as long as $|u[i]| < \epsilon$ for some ϵ, that the solution $x[i]$ will be bounded for all time t.**

(HINT: There are a few helpful facts about absolute values and inequalities that are helpful in such proofs. First: $\left|\sum_j a_j\right| \leq \sum_j |a_j|$. Second $|ab| = |a| \cdot |b|$. Third: $\left|e^{j\theta}\right| = 1$ no matter what real number θ is. And fourth, if $a_i > 0$ and $b_i > 0$, and $b_i \leq B$, then $\sum_i a_i b_i \leq \sum_i a_i B = B \sum a_i$.)

4. **Eigenvalue Placement through State Feedback**

   Consider the following discrete-time linear system:

   $$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]. \tag{5}$$

   In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$.

   (a) **Is this system controllable?**

   (b) **Is this discrete-time linear system stable in open loop (without feedback control)?**

   (c) Suppose we use state feedback of the form $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] = F\vec{x}[i]$.

   Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

   (d) We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrices A and B from (a) for the system $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i]$ into the Jupyter notebook "`eigenvalue_placement.ipynb`" and use the randomly generated $\vec{w}[i]$ as the disturbance introduced into the state equation. Observe how the norm of $\vec{x}[i]$ evolves over time for the given A. **What do you see happening to the norm of the state?**

   (e) Add the feedback computed in part (c) to the system in the notebook and **explain how the norm of the state changes.**

   (f) Now suppose we've got a different system described by the controlled scalar difference equation $z[i+1] = z[i] + 2z[i-1] + u[i]$. To convert this second-order difference equation to a two-dimensional discrete time system, we will let $\vec{y}[i] = \begin{bmatrix} z[i-1] \\ z[i] \end{bmatrix}$.

   Write down the system representation for $\vec{y}$ in the following matrix form:

   $$\vec{y}[i+1] = A_y\vec{y}[i] + B_y u[i]. \tag{6}$$

   Specify the values of the matrix A_y and the vector B_y.

   (g) It turns out that the original $\vec{x}[i]$ system can be converted to the $\vec{y}[i]$ system using a change of basis P. Let this coordinate change be written as $\vec{y}[i] = P\vec{x}[i]$. **First express A_y and B_y symbolically in terms of A, B, and P. Then, confirm numerically that $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ is the correct change of basis matrix between the two systems.**

   (h) For the $\vec{y}$ system from part (f), **design a feedback gain matrix $\begin{bmatrix} \overline{f}_1 & \overline{f}_2 \end{bmatrix}$ to place the closed-loop eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Additionally, confirm that this matrix is just a change of basis of the gain matrix from part (c), i.e. $\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \overline{f}_1 & \overline{f}_2 \end{bmatrix} P$.**

   Note that this means you can solve for the closed-loop gains of your system in any basis, and then transform it to the basis you care about.

5. **Tracking a Desired Trajectory in Continuous Time**

The treatment in 16B so far has treated closed-loop control as being about holding a system steady at some desired operating point, by placing the eigenvalues of the state transition matrix. This control used the actual current state to apply a control signal designed to bring the eigenvalues in the region of stability. Meanwhile, the idea of controllability itself was more general and allowed us to make an open-loop trajectory that went pretty much anywhere. This problem is about combining these two ideas together to make feedback control more practical — how we can get a system to more-or-less closely follow a desired trajectory, even though it might not start exactly where we wanted to start and in principle could be affected by small disturbances throughout.

In this question, we will also see that everything that you have learned to do closed-loop control in discrete-time can also be used to do closed-loop control in continuous time.

Consider the specific 2-dimensional system

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) + \vec{w}(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t) + \vec{w}(t) \tag{7}$$

where $u(t)$ is a scalar valued continuous control input and $\vec{w}(t)$ is a bounded disturbance (noise).

(a) In an ideal noiseless scenario, the desired control signal $u^*(t)$ makes the system follow the desired trajectory $\vec{x^*}(t)$ that satisfies the following dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x^*}(t) = A\vec{x^*}(t) + \vec{b}u^*(t) \tag{8}$$

The presence of the bounded noise term $\vec{w}(t)$ makes the actual state $\vec{x}(t)$ deviate from the desired $\vec{x^*}(t)$ and follow (7) instead. In the following subparts, we will analyze how we can adjust the desired control signal $u^*(t)$ in (8) to the control input $u(t)$ in (7) so that the deviation in the state caused by $\vec{w}(t)$ remains bounded.

Represent the state as $\vec{x}(t) = \vec{x^*}(t) + \vec{v}(t)$ and $u(t) = u^*(t) + u_v(t)$. Using (7) and (8), we can represent the evolution of the trajectory deviation $\vec{v}(t)$ as a function of the control deviation $u_v(t)$ and the bounded disturbance $\vec{w}(t)$ as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t) = A_v\vec{v}(t) + \vec{b}_v u_v(t) + \vec{w}(t) \tag{9}$$

What are A_v and $\vec{b}_v$ in terms of the original system parameters A and $\vec{b}$? (*HINT: Write out equation* (7) *in terms of* $\vec{x^*}(t)$, $\vec{v}(t)$, $u^*(t)$ *and* $u_v(t)$.)

(b) **Are the dynamics that you found for $\vec{v}(t)$ in part (b) stable? Based on this, in the presence of bounded disturbance $\vec{w}(t)$, will $\vec{x}(t)$ in (7) follow the desired trajectory $\vec{x^*}(t)$ closely if we just apply the control $u(t) = u^*(t)$ to the original system in (7), i.e. $u_v(t) = 0$?**
(*HINT: Use the numerical values of A and $\vec{b}$ from* (7) *in the solution from part (b) to determine stability of $\vec{v}(t)$.*)

Now, we want to apply state feedback control to the system using $u_v(t)$ to get our system to follow the desired trajectory $\vec{x^*}(t)$.

(c) For the $\vec{v}(t), u_v(t)$ system, **apply feedback control by letting $u_v(t) = F\vec{v}(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix}\vec{v}(t)$ that would place both the eigenvalues of the closed-loop $\vec{v}(t)$ system at -10. Find f_0 and f_1.**

(d) Based on what you did in the previous parts, and given access to the desired trajectory $\vec{x^*}(t)$, the desired control $u^*(t)$, and the actual measurement of the state $\vec{x}(t)$, **come up with a way to do feedback control that will keep the trajectory staying close to the desired trajectory no matter what the small bounded disturbance $\vec{w}(t)$ does.** *(HINT: Express the control input $u(t)$ in terms of $u^*(t)$, $\vec{x^*}(t)$, and $\vec{x}(t)$.)*

6. Miscellaneous Practice Problems for Midterm

(a) You are given the graph in Figure 2. **Express the coordinates of vectors $\vec{v}$ and $\vec{w}$ in both Cartesian (x, y) and Polar $\left(re^{j\theta}\right)$ forms**.

You may use the $\text{atan2}()$ or $\tan^{-1}$ function for angle (θ) as necessary.
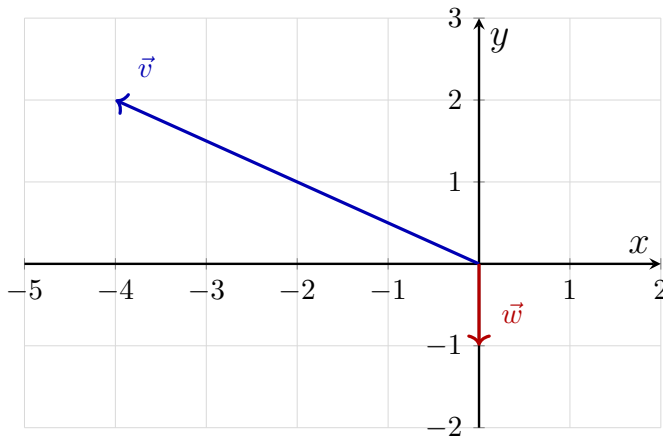
Figure 2: Vectors in the $\mathbf{x} - \mathbf{y}$ plane

i. Label $\vec{v}$ with its corresponding Cartesian (x, y) and Polar $\left(re^{j\theta}\right)$ coordinates, in the given form.

ii. Label $\vec{w}$ with its corresponding Cartesian (x, y) and Polar $\left(re^{j\theta}\right)$ coordinates, in the given form.

(b) You are given an input voltage signal below:

$$v_{\text{in}}(t) = -1.5 \sin\left(\omega t - \frac{\pi}{3}\right). \tag{10}$$

Convert the signal of eq. (10) to its phasor representation. That is, find $\widetilde{V}_{\text{in}}$. *Justify your answer.*

(c) You decided to analyze the transfer function of a band-pass filter, and have generated the following Bode plots for $H(\mathrm{j}\omega)$. **If your input voltage signal is**

$$v_{\text{in}}(t) = 4\cos\left(\omega_s t + \frac{2\pi}{3}\right), \tag{11}$$

where $\omega_s = 1 \times 10^4 \frac{\text{rad}}{\text{s}}$, **what is the approximate value of** $v_{\text{out}}(t)$ **based on the Bode plots**? *Since the original transfer function is not provided, you cannot numerically compute the exact values of magnitude and phase. Just read the approximate values from the Bode plot.*
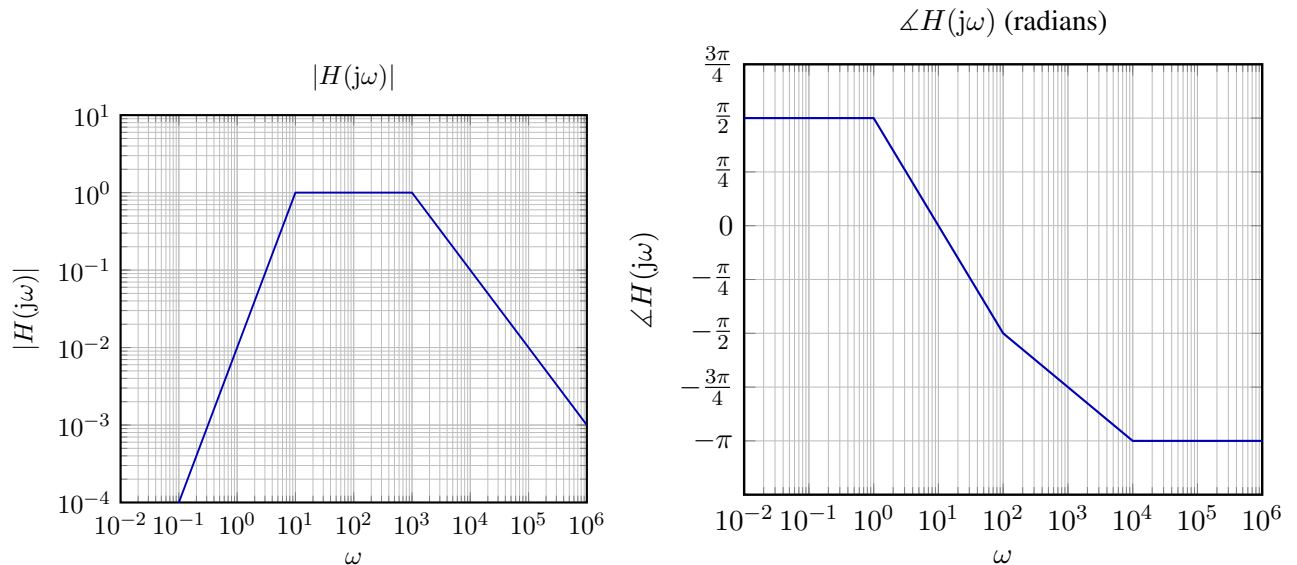

(d) Assume that the overall transfer function of a new filter, $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$, is given by

$$H(\mathrm{j}\omega) = \left(\frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_{c1}}} + \frac{\mathrm{j}\frac{\omega}{\omega_{c2}}}{1 + \mathrm{j}\frac{\omega}{\omega_{c2}}}\right), \tag{12}$$

where $\omega_{c2} = 100\omega_{c1}$. **Qualitatively describe the magnitude of the transfer function** $|H(\mathrm{j}\omega)|$ **in three regions: frequencies below** ω_{c1}**, frequencies between** ω_{c1} **and** ω_{c2}**, and frequencies above** ω_{c2}**. Identify the filter type by explaining what it is doing qualitatively** (for example, a low-pass filter passes low frequencies but does not pass high frequencies).

7. [Optional] Op-Amp Practice Problem

We are going to analyze the following op-amp circuit in the phasor domain. All the voltages and currents in the problem are phasors and all the Z_i are impedances.
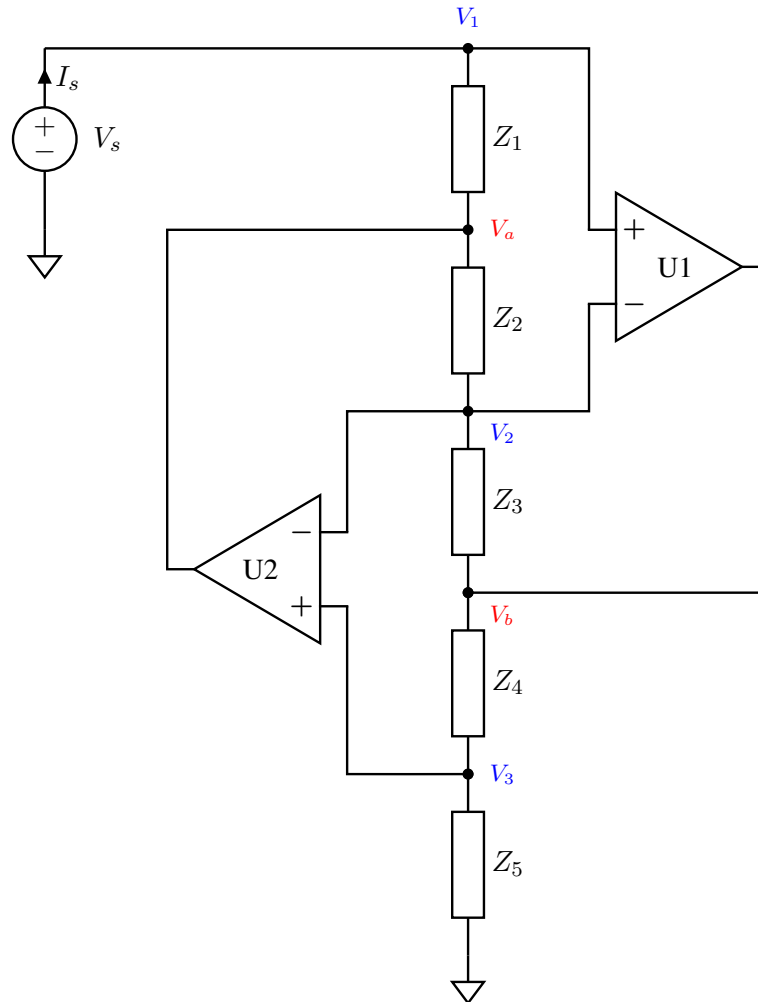
(a) Treat all the Op-Amps as being in negative feedback and therefore following the Golden Rules. **What are the voltages at V_1, V_2, and V_3 in terms of V_s?**

(b) **Express I_s in terms of V_s, V_a, Z_1.**

(c) The impedance Z_s is defined as $\left(Z_s = \frac{V_s}{I_s} \right)$. **Find Z_s in terms of Z_1, Z_2, Z_3, Z_4, Z_5.**

(d) Assume the following:

$$Z_1 = R_1 \tag{13}$$

$$Z_2 = \frac{1}{j\omega C_2} \tag{14}$$

$$Z_3 = R_3 \tag{15}$$

$$Z_4 = R_4 \tag{16}$$

$$Z_5 = R_5 \tag{17}$$

Evaluate Z_s for the above case.

(e) **Is Z_s inductive or capacitive? If it is inductive, find its inductance. If it is capacitive, find its capacitance.**

8. **Write Your Own Question And Provide a Thorough Solution.**

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning (from the bottom up) is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

You need to write your own question and provide a thorough solution to it. The scope of your question should roughly overlap with the scope of this entire problem set. This is because we want you to exercise your understanding of this material, and not earlier material in the course. However, feel free to combine material here with earlier material, and clearly, you don't have to engage with everything all at once. A problem that just hits one aspect is also fine.

Note: One of the easiest ways to make your own problem is to modify an existing one. Ordinarily, we do not ask you to cite official course materials themselves as you solve problems. This is an exception. Because the problem making process involves creative inputs, you should be citing those here. It is a part of professionalism to give appropriate attribution.

Just FYI: Another easy way to make your own question is to create a Jupyter part for a problem that had no Jupyter part given, or to add additional Jupyter parts to an existing problem with Jupyter parts. This often helps you learn, especially in case you have a programming bent.

9. **Homework Process and Study Group**

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

   (a) **What sources (if any) did you use as you worked through the homework?**

   (b) **If you worked with someone on this homework, who did you work with?**
   List names and student ID's. (In case of homework party, you can also just describe the group.)

   (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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