EECS 16B Designing Information Devices and Systems II
Fall 2021 UC Berkeley

Homework 14

This homework is due on Thursday, December 9, 2021, at 11:59PM. Self-grades and HW Resubmission are due on Saturday, December 11, 2021, at 11:59PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 14 Note 15 Note 19 Note 2j

   (a) **What is the least squares solution for the problem $A\vec{x} \approx \vec{b}$ when all matrices and vectors have complex entries?**

   (b) **Given a complex orthonormal matrix A, what is the projection of a vector $\vec{b}$ onto the columns of A?**

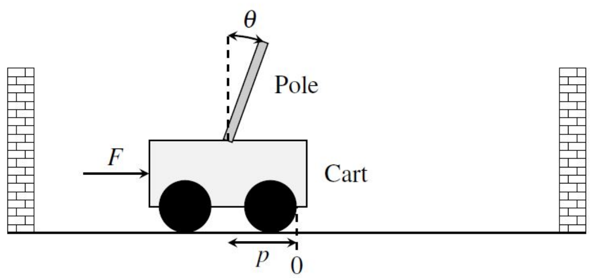
   (c) **What is the pseudoinverse of a matrix A in terms of the compact SVD of A?** You may find looking at a previous homework problem helpful.

2. Segway Tours (Optional)

A segway is a stand on two wheels, and can be thought of as an inverted pendulum. The segway works by applying a force (through the spinning wheels) to the base of the segway, This controls both the position on the segway and the angle of the stand. As the driver pushes on the stand, the segway tries to bring itself back to the upright position, and it can only do this by moving the base.

Recall that you have analyzed a basic version of this segway question in Homework 0 problem 5. You were given a linear discrete time representation of the segway dynamics, and were guided through the steps to find if it's possible to make the segway reach some desired states, essentially laying the foundation of controllability. Now, we will see how to derive the linear discrete time system from the equations of motion, and then do some further refined analysis based on our improved knowledge of controllabilty.

The main question we wish to answer is: Is it possible for the segway to be brought upright and to a stop from any initial configuration? There is only one input (force) used to control two outputs (position and angle). Let's model the segway as a cart-pole system and analyze.



A cart-pole system can be fully described by its position p, velocity $\frac{\mathrm{d}p}{\mathrm{d}t}$, angle θ, and angular velocity $\frac{\mathrm{d}\theta}{\mathrm{d}t}$. We can write this as the continuous time state vector $\vec{x}$ as follows:

$$\vec{x} = \begin{bmatrix} p \\ \frac{\mathrm{d}p}{\mathrm{d}t} \\ \theta \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} \end{bmatrix} \tag{1}$$

The input to this system is a scalar quantity $u(t)$ at time t, which is the force F applied to the cart (or base of the segway). Let the coefficient of friction be k.

The equations of motion for this system are as follows:

$$\begin{aligned} \frac{\mathrm{d}^2 p}{\mathrm{d}t^2} &= \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 l\sin\theta - g\sin\theta\cos\theta - \frac{k}{m}\frac{\mathrm{d}p}{\mathrm{d}t}\right) \\ \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} &= \frac{1}{l\left(\frac{M}{m} + \sin^2\theta\right)}\left(-\frac{u}{m}\cos\theta - \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 l\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta + \frac{k}{m}\frac{\mathrm{d}p}{\mathrm{d}t}\cos\theta\right) \end{aligned} \tag{2}$$

The derivation of these equations is a mechanics problem and not in 16B scope, but interested students can look up the details online.

(a) First let's linearize the system of equations in (2) about the upright position at rest, i.e. $\theta_* = 0$ and

$\frac{d\theta}{dt}_* = 0$. **Show that the linearized system of equations is given by the following state space form:**

$$\frac{d\vec{x}(t)}{dt} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{k}{M} & -\frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{k}{Ml} & \frac{M+m}{Ml}g & 0 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix}}_{\vec{b}} u(t) \tag{3}$$

(HINT: Since we are linearizing around $\theta_* = 0$ and $\frac{d\theta}{dt}_* = 0$, you can use the following approximations for small values of θ:

$$\sin\theta \approx \theta$$
$$\sin^2\theta \approx 0$$
$$\cos\theta \approx 1$$
$$\left(\frac{d\theta}{dt}\right)^2 \approx 0.$$

You do not have to do the full linearization using Taylor series, you can just substitute the approximations above. You will get the same answer as doing the linear Taylor series approximation.)

Notice that for this particular choice of θ_* and $\frac{d\theta}{dt}_*$, the linearization does not depend on what p or $\frac{dp}{dt}$ is. This is partially a stroke of luck and partially a consequence of the fact that the position p doesn't appear in the dynamics equations.

(b) For all subsequent parts, assume that $m = 1$, $M = 10$, $g = 10$, $l = 1$ and $k = 0.1$. Let's consider the discrete time representation of the state space (3) at time $t = n\Delta$. For simplicity, assume $\Delta = 1$. The discrete time state $\vec{x}_d$ follows the following linear model:

$$\vec{x}_d[n+1] = A_d\vec{x}_d[n] + \vec{b}_d u_d[n] \tag{4}$$

where $A_d \in \mathbb{R}^{4\times4}$ and $\vec{b}_d \in \mathbb{R}^{4\times1}$. **Find A_d and $\vec{b}_d$ in terms of the eigenvalues and eigenvectors of A, and Δ. State numerical values for A_d and $\vec{b}_d$.**. Use the Jupyter notebook `segway.ipynb` for all numerical calculations, and approximate the results to 2 or 3 significant figures.

(HINT: Recall that the continuous time scalar differential equation

$$\frac{dz(t)}{dt} = \lambda z(t) + cw(t)$$

can be represented in discrete time ($n\Delta = t$) as follows:

$$z_d[n+1] = \begin{cases} (e^{\lambda\Delta}) \cdot z_d[n] + \left(\frac{e^{\lambda\Delta}-1}{\lambda}\right) \cdot cw_d[n] & \text{if } \lambda \neq 0 \\ (1) \cdot z_d[n] + (\Delta) \cdot cw_d[n] & \text{if } \lambda = 0 \end{cases}$$

Use the eigendecompostion of $A = V\Lambda V^{-1}$ to do change of basis variables, and you should finally reach

$$\vec{x}_d[n+1] = \underbrace{V\Lambda_d V^{-1}}_{A_d}\vec{x}_d[n] + \underbrace{VM_dV^{-1}\vec{b}}_{\vec{b}_d}u_d[n]$$

What are the elements of Λ_d and M_d in terms of the elements of Λ? You may find in later parts of the notebook that you have A_d and $\vec{b}_d$ which can serve as a sanity check for your derivation and numerical calculations.)

(c) **Show that the linear-approximation discrete time system in (4) is controllable by using the appropriate matrix in the Jupyter notebook.**

(HINT: Is the controllability matrix full rank? You have to use numerical values of A_d and $\vec{b}_d$ from the previous part. Use the Jupyter notebook for all numerical calculations.)

(d) Since the linear-approximation discrete time system is controllable, it is possible to reach any final state $\vec{x}_{d,f}$ starting from any initial state $\vec{x}_{d,i}$ using an appropriate sequence of inputs in exactly 4 steps, provided that the deviations are small enough so that the linearization approximation is valid. **Set up a set of linear equations to solve for the $u_d[0]$, $u_d[1]$, $u_d[2]$, $u_d[3]$ given the initial and final states.**

Find the input sequence to reach the upright position $\vec{x}_{d,f} = \vec{x}_d[4] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ starting from an initial state $\vec{x}_{d,i} = \vec{x}_d[0] = \begin{bmatrix} -2 \\ 3.1 \\ 0.3 \\ -0.6 \end{bmatrix}$. Use the Jupyter notebook for all numerical calculations and simulation. **Explain qualitatively what you observe from the segway simulation.**

(HINT: Use (4) and loop unrolling to express $\vec{x}_d[4]$ as a linear combination of $\vec{x}_d[0]$, $u_d[3]$, $u_d[2]$, $u_d[1]$, $u_d[0]$.)

(e) Now suppose we try to use an initial state $\vec{x}_{d,i} = \vec{x}_d[0] = \begin{bmatrix} -2 \\ 3.1 \\ 3.3 \\ -0.6 \end{bmatrix}$ for which the approximation is poor since $\theta_i = 3.3$ is very far from the linearization point $\theta_* = 0$. **Using the equations derived in the previous part, use the Jupyter notebook to determine the input sequence to reach the same final upright position. Explain qualitatively what you observe from the segway simulation.** Use the Jupyter notebook for all numerical calculations and simulation.

Compare the simulation results in parts (d) and (e). In both cases, the segway finally stabilizes to an upright position at rest. However, in part (d) the behavior of the segway looks more realistic whereas in part (e) it is doing some wild unexpected rotations.

This is because the linearization approximation is valid with the small initial values of θ and $\frac{d\theta}{dt}$ in part (d). So this discrete time linear model is a good representation of the original continuous time non-linear system. Hence the trajectory taken by the segway from the initial to the final position is similar to what we may expect from real life physics.

However in part (e), the linearization approximation is not really valid. The approximate model still converges to the final upright position because (4) is controllable as we proved in part (c). However, since the approximation is not valid anymore, this discrete time linear model is **not** a good representation of the original continuous time non-linear system. Hence the predicted trajectory is extremely weird with the segway undergoing a few full rotations, and does not match what we would expect from the real system.

We can still analyze the system in continuous time by directly solving the set of non-linear differential equations in (2) (out of 16B scope) or in discrete time using a finely discretized (and still nonlinear) version of (2). Note that there are two independent distinctions we are making, i.e. continuous vs discrete, and linear vs non-linear. Part (e) failed because it's beyond the scope of the linear model, not because we are using a discrete time system. A non-linear discrete time analysis would also give the correct solution.

(f) Let's analyze the behavior of the segway by comparing the continuous time linear model and continuous time non-linear model. Deriving the control input to bring the segway to the upright position at rest requires more care which is out of 16B scope, so we will just look at the simple case of the segway freely settling to steady state in the absence of any control input, i.e. $u(t) = 0$. **Toggle the `linearized` flag between `True` and `False` in the Jupyter notebook, and qualitatively explain the differences in the trajectory as the segway freely swings around.**

There's actually much more that we could have you do with this problem with what is in scope in 16B. But the semester is drawing to a close and you need to study for other courses too. So we will stop here.

3. Adapting Proofs to the Complex Case

At many points in the course, we have made assumptions that various matrices or eigenvalues are real while discussing various theorems. If you have noticed, this has always happened in contexts where we have invoked orthonormality during the proof or statement of the result. Now that you understand the idea of orthonormality for complex vectors, and how to adapt Gram-Schmitt to complex vectors, you can go back and remove those restrictions. This problem asks you to do exactly that.

Unlike many of the problems that we have given you in 16A and 16B, this problem has far less hand-holding — there aren't multiple parts breaking down each step for you. Fortunately, you have the existing proofs in your notes to work based on. So this problem has a secondary function to help you solidify your understanding of these earlier concepts ahead of the final exam.

There is one concept that you will need beyond the idea of what orthogonality means for complex vectors as well as the idea of conjugate-transposes of vectors and matrices. The analogy of a real symmetric matrix S that satisfies $S = S^\top$ is what is called a Hermitian matrix H that satisfies $H = H^*$ where $H^* = \overline{H^\top}$ is the conjugate-transpose of H.

(a) The upper-triangularization theorem for all (potentially complex) square matrices A says that there exists an orthonormal (possibly complex) basis V so that V^*AV is upper-triangular.

Adapt the proof from the real case with assumed real eigenvalues to prove this theorem.

Feel free to assume that any square matrix has an (potentially complex) eigenvalue/eigenvector pair. You don't need to prove this. But you can make no other assumptions.

(HINT: Use the exact same argument as before, just use conjugate-transposes instead of transposes.)

Congratulations, once you have completed this part you essentially can solve all systems of linear differential equations based on what you know, and you can also complete the proof that having all the eigenvalues being stable implies BIBO stability.

(b) The spectral theorem for Hermitian matrices says that a Hermitian matrix has all real eigenvalues and an orthonormal set of eigenvectors.

Adapt the proof from the real symmetric case to prove this theorem.

(HINT: As before, you should just leverage upper-triangularization and use the fact that $(ABC)^* = C^*B^*A^*$. There is a reason that this part comes after the first part.)

(c) The SVD for complex matrices says that any rectangular (potentially complex) matrix A can be written as $A = \sum_i \sigma_i \vec{u}_i \vec{v}_i^*$ where σ_i are real positive numbers and the collection $\{\vec{u}_i\}$ are orthonormal (but potentially complex) as well as $\{\vec{v}_i\}$.

Adapt the derivation of the SVD from the real case to prove this theorem.

Feel free to assume that A is wide. (Since you can just conjugate-transpose everything to get a tall matrix to become wide.)

(HINT: Analogously to before, you're going to have to show that the matrix A^*A is Hermitian and that it has non-negative eigenvalues. Use the previous part. There is a reason that this part comes after the previous parts.)

4. Minimum Norm Variants (Optional)

In lecture and HW, you saw how to solve minimum norm problems in which we have a wide matrix A and solve $A\vec{x} = \vec{y}$ such that $\vec{x}$ is a minimum norm solution: $\|\vec{x}\| \leq \|\vec{z}\|$ for all $\vec{z}$ such that $A\vec{z} = \vec{y}$.

We also saw in the HW how we can solve some variants in which we were interested in minimizing the norm $\|C\vec{x}\|$ instead. You have solved the case where C is invertible and square or a tall matrix. This question asks you about the case when C is a wide matrix. The key issue is that wide matrices have nontrivial nullspaces — that means that there are "free" directions in which we can vary $\vec{x}$ while not having to pay anything. How do we best take advantage of these "free" directions?

Parts (a-b) are connected; parts (c-d) are another group that can be done independently of (a-b); and parts (e-g) are another group that can be started independently of either (a-b) or (c-d). If you get stuck, try another group. ***During debugging, many TAs found it easier to start with parts (c-g), and coming back to (a-b) at the end.***

In parts (a) and (b) you will reduce the problem of minimizing $\|C\vec{x}\|$ to a problem which you solved a variant of: minimizing only the norm of a vector $\|\vec{\widetilde{x}}_c\|$ with some constraint involving some freely chosen $\vec{\widetilde{x}}_f$ without requiring direct consideration of the matrix C. With (a) and (b), we wish to give you another example of converting problems that are new into problems you have solved before or problems that are proximal to ones you have solved before. In parts (c) and (d) we emphasize the skill of solving a problem in a nice setting to gain some insight before relaxing the nice properties in parts (e) to (g) for more generality.

(a) Given a wide matrix A (with m columns and n rows) and a wide matrix C (with m columns and r rows), we want to solve:

$$\min_{\vec{x} \text{ such that } A\vec{x}=\vec{y}} \|C\vec{x}\| \tag{5}$$

As mentioned above, the key new issue is to isolate the "free" directions in which we can vary $\vec{x}$ so that they might be properly exploited. Consider the full SVD of $C = U\Sigma_C V^\top = \sum_{i=1}^{\ell} \sigma_{c,i}\vec{u}_i\vec{v}_i^\top$. Here, we write:

$$V = \begin{bmatrix} V_C & | & V_F \end{bmatrix}, \quad V_C = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_\ell \\ | & | & & | \end{bmatrix}, V_F = \begin{bmatrix} | & & | \\ \vec{v}_{\ell+1} & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \tag{6}$$

so that the columns of V_C all correspond to singular values $\sigma_{c,i} > 0$ of C, and the columns of V_F form an orthonormal basis for the nullspace of C.

Change variables in the problem to be in terms of $\vec{\widetilde{x}} = \begin{bmatrix} \vec{\widetilde{x}}_c \\ \vec{\widetilde{x}}_f \end{bmatrix}$ where the ℓ-dimensional $\vec{\widetilde{x}}_c$ has i-th entry $\widetilde{x}_{c,i} = \alpha_i \vec{v}_i^\top \vec{x}$, and the $(m-\ell)$-dimensional $\vec{\widetilde{x}}_f$ has i-th entry $\widetilde{x}_{f,i} = \vec{v}_{\ell+i}^\top \vec{x}$. In vector/matrix form,

$\vec{\tilde{x}}_f = V_F^\top \vec{x}$ and $\vec{\tilde{x}}_c = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_\ell \end{bmatrix} V_C^\top \vec{x}$. Or directly:

$$\vec{\tilde{x}} = \begin{bmatrix} \vec{\tilde{x}}_c \\ \vec{\tilde{x}}_f \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_\ell \end{bmatrix} V_C^\top \\ V_F^\top \end{bmatrix} \vec{x}, \quad \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_\ell \end{bmatrix} V_C^\top \in \mathbb{R}^{\ell \times m}, \quad V_F^\top \in \mathbb{R}^{(m-\ell)\times m}. \tag{7}$$

Express $\vec{x}$ in terms of $\vec{\tilde{x}}_f$ and $\vec{\tilde{x}}_c$. Assume the $\alpha_i \neq 0$ so the relevant matrix is invertible.

What is $\|C\vec{x}\|$ in terms of $\vec{\tilde{x}}_f$ and $\vec{\tilde{x}}_c$? Simplify as much as you can for full credit.

(HINT: If you get stuck on how to express $\vec{x}$ in terms of the new variables, think about the special case when $\ell = 1$ and $\alpha_1 = \frac{1}{2}$. How is this different from when $\alpha_1 = 1$? The SVD of C might be useful when looking at $\|C\vec{x}\|$. A fact you may use without proof is that a vector $\vec{x}$ may be decomposed uniquely as $\vec{x} = \vec{x}_V + \vec{x}_{V\perp}$ where $\vec{x}_V$ is in the span of a matrix V's columns, and $\vec{x}_{V\perp}$ is in the subspace of vectors orthogonal to V's columns.)

(b) Continuing the previous part, **give appropriate values for the α_i so that the problem (5) becomes**

$$\min_{\vec{\tilde{x}} = \begin{bmatrix} \vec{\tilde{x}}_c \\ \vec{\tilde{x}}_f \end{bmatrix} \text{ such that } \begin{bmatrix} A_C & | & A_F \end{bmatrix} \begin{bmatrix} \vec{\tilde{x}}_c \\ \vec{\tilde{x}}_f \end{bmatrix} = \vec{y}} \|\vec{\tilde{x}}_c\| \tag{8}$$

Give explicit expressions for A_C and A_F in terms of the original A and terms arising from the SVD of C. Because you have picked values for the α_i, there should be no α_i in your final expressions for full credit.

(HINT: How do the singular values $\sigma_{c,i}$ interact with the α_i? Then apply the appropriate substitution to (5) to get (8).)

(c) Let us focus on a simple case. *(You can do this even if you didn't get the previous parts.)* Suppose that $A = \begin{bmatrix} A_C & | & A_F \end{bmatrix}$ where the columns of A_F are orthonormal, as well as orthogonal to the columns of A_C. The columns of A together span the entire n-dimensional space. We directly write $\vec{x} = \begin{bmatrix} \vec{x}_c \\ \vec{x}_f \end{bmatrix}$ so that $A\vec{x} = A_C\vec{x}_c + A_F\vec{x}_f$. Now suppose that we want to solve $A\vec{x} = \vec{y}$ and only care about minimizing $\|\vec{x}_c\|$. We don't care about the length of $\vec{x}_f$ — it can be as big or small as necessary. In other words, we want to solve:

$$\min_{\vec{x} = \begin{bmatrix} \vec{x}_c \\ \vec{x}_f \end{bmatrix} \text{ such that } \begin{bmatrix} A_C & | & A_F \end{bmatrix} \begin{bmatrix} \vec{x}_c \\ \vec{x}_f \end{bmatrix} = \vec{y}} \|\vec{x}_c\| \tag{9}$$

Show that the optimal solution has $\vec{x}_f = A_F^\top \vec{y}$.

(HINT: Multiplying both sides of something by $A_F^\top$ might be helpful.)

(d) Continuing the previous part, **compute the optimal $\vec{x}_c$.** Show your work.

(HINT: What is the work that $\vec{x}_c$ needs to do? $\vec{y} - A_F A_F^\top \vec{y}$ might play a useful role, as will the SVD of $A_C = \sum_i \sigma_i \vec{t}_i \vec{w}_i^\top$.)

(e) Now suppose that A_C did not necessarily have its columns orthogonal to A_F. Continue to assume that A_F has orthonormal columns. *(You can do this part even if you didn't get any of the previous parts.)* Write the matrix $A_C = A_{C\perp} + A_{CF}$ where the columns of A_{CF} are all in the column span of A_F and the columns of $A_{C\perp}$ are all orthogonal to the columns of A_F. **Give an expression for A_{CF} in terms of A_C and A_F.**

(HINT: What does this have to do with projection and least squares?)

(f) Continuing the previous part, **compute the optimal $\vec{x}_c$** that solves (9): (copied below)

$$\min_{\vec{x}=\begin{bmatrix}\vec{x}_c\\\vec{x}_f\end{bmatrix} \text{ such that } \begin{bmatrix} A_C & | & A_F \end{bmatrix}\begin{bmatrix}\vec{x}_c\\\vec{x}_f\end{bmatrix}=\vec{y}} \|\vec{x}_c\|$$

Show your work. Feel free to call the SVD as a black box as a part of your computation.

(HINT: What is the work that $\vec{x}_c$ needs to do? The SVD of $A_{C\perp}$ might be useful.)

(g) Continuing the previous part, **compute the optimal $\vec{x}_f$.** Show your work.

You can use the optimal $\vec{x}_c$ in your expression just assuming that you did the previous part correctly, even if you didn't. You can also assume a decomposition $A_C = A_{C\perp} + A_{CF}$ from further above in part (e) without having to write what these are, just assume that you did them correctly, even if you didn't do them at all.

(HINT: What is the work that $\vec{x}_f$ needs to do? How is A_{CF} relevant here?

5. Remote Final Accomodation

If you are a student who will be outside of the 150 mile radius of UC Berkeley or if you are a student who has a University-approved accommodation to being in-person, such as having an exemption to the vaccine mandate, please fill out this Google form if you haven't already to request remote accommodations for the final exam which will take place on Friday, December 17, from 8 to 11 am PT.

6. Study Group Feedback Survey

If you are a student who participated in the study group survey we gave in the early weeks of the semester, we would really appreciate your feedback on the group you were matched with. Though we had a section devoted to it in the mid-semester feedback form, it was anonymized, which can not serve as good data for the research team. Please fill out this Google form to provide feedback about the study groups.

7. End of Semester Survey (Updated)

Congratulations on making it to the last week of the semester! We would love to get your feedback about the class. Please fill out this Google form. We will be awarding 1 global extra credit point to every student in the class if we hit 70% end-of-semester survey completion by each individual grade category in this class by Wednesday, December 15, at 11:59 PM. In other words, we want at least 70% of freshmen, 70% of sophomores, 70% of juniors, and 70% of seniors in this class to complete the survey in order for everyone to get extra credit. To improve our ability to zoom in on helping students who are struggling, we are asking for students to voluntarily disclose their identities while filling out the surveys (unlike the campus course surveys, we do not store your identities with survey responses). Such disclosures are purely voluntary, and our only purpose in asking for them is to be able to zoom in on the much smaller subpopulation that is performing poorly in the course and see what might be different for them.

8. Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very important way to really learn material. The famous "Bloom's Taxonomy" that lists the levels of learning (from the bottom up) is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don't want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don't have to achieve this every week. But unless you try every week, it probably won't ever happen.

You need to write your own question and provide a thorough solution to it. The scope of your question should roughly overlap with the scope of this entire problem set. This is because we want you to exercise your understanding of this material, and not earlier material in the course. However, feel free to combine material here with earlier material, and clearly, you don't have to engage with everything all at once. A problem that just hits one aspect is also fine.

Note: One of the easiest ways to make your own problem is to modify an existing one. Ordinarily, we do not ask you to cite official course materials themselves as you solve problems. This is an exception. Because the problem making process involves creative inputs, you should be citing those here. It is a part of professionalism to give appropriate attribution.

Just FYI: Another easy way to make your own question is to create a Jupyter part for a problem that had no Jupyter part given, or to add additional Jupyter parts to an existing problem with Jupyter parts. This often helps you learn, especially in case you have a programming bent.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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