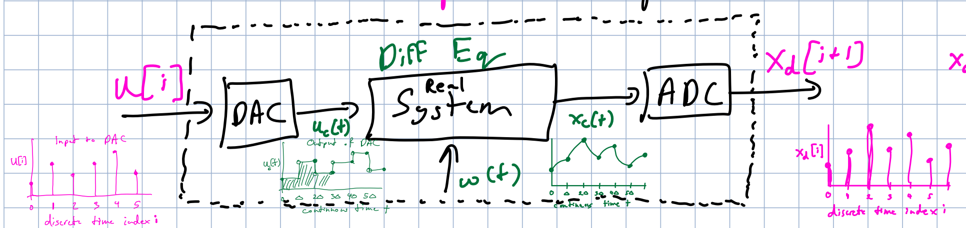
16B Prof. Anant Sahai

Reminder: In-Person Midterm Mon Oct 18
7-9 pm
Locations: TBA
Scope: Through HW 7 due Friday Oct 15th

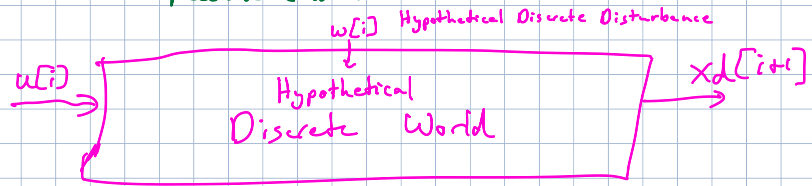
Today: Finish system identification (Note 9)
Control and Stability (Note 10)

How does a computer experience the world?




$u_c(t) = u[i]$ if $t \in (i\Delta, (i+1)\Delta)$

↖ piecewise-constant




Discrete model: $x_d[i+1] = a\, x_d[i] + b\, u[i] + w[i]$

Vector case: $\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] + \vec{w}[t]$

$$A = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix} \qquad B = \begin{bmatrix} \vec{b}_1^T \\ \vec{b}_2^T \\ \vdots \\ \vec{b}_n^T \end{bmatrix}$$

From Last Lecture

Look by rows of unknowns A and B

Stack unknowns vertically

$$T\left\{ \begin{bmatrix} \vec{x}[0]^T & \vec{u}[0]^T \\ \vdots & \vdots \\ \vec{x}[T-1]^T & \vec{u}[T-1]^T \end{bmatrix} \right. \begin{array}{c} \begin{bmatrix} \vec{a}_r \\ \vec{b}_r \end{bmatrix} \end{array} \begin{array}{c} n \\ m \end{array} \approx \begin{bmatrix} x_r[1] \\ x_r[2] \\ \vdots \\ x_r[T] \end{bmatrix}$$

$n + m$

tall matrix $T\{\overbrace{D}^{n+m}$ $\qquad \vec{p}_r\}^{n+m}$ $\qquad \vec{s}_r\}\, T$

$$D\vec{p}_r \approx \vec{s}_r$$

Least Squares $\hat{\vec{p}}_r = \underbrace{(D^T D)^{-1} D^T}_{\underline{\text{No r's here!}}} \vec{s}_r$

$$\hat{p}_1 = \underbrace{(D^T D)^{-1} D^T}_{M} \vec{s}_1$$

$$\hat{p}_2 = M \vec{s}_2$$

$$\vdots$$

$$\vec{p}_n = M \vec{s}_n$$

$$\underbrace{\begin{bmatrix} \hat{p}_1 & \hat{p}_2 & \cdots & \hat{p}_n \end{bmatrix}}_{\hat{P}} = M \underbrace{\begin{bmatrix} \vec{s}_1 & \vec{s}_2 & \cdots & \vec{s}_n \end{bmatrix}}_{S}$$

$$\hat{P} = \begin{bmatrix} \hat{A}^T \\ \hat{B}^T \end{bmatrix}$$

$$S = \begin{bmatrix} x_1[1] & x_2[1] & \cdots & x_n[1] \\ \vdots & & \ddots & \vdots \\ x_1[T] & \cdots & & x_n[T] \end{bmatrix} = \begin{bmatrix} \vec{x}[1]^T \\ \vdots \\ \vec{x}[T]^T \end{bmatrix}$$

$$\hat{P} = (D^T D)^{-1} D^T S$$

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We have $\hat{P}$, how do we know if it is any good?

<u>Gold Standard</u>: Use the $\hat{P}$ for control and see if it works?

Silver Standard: Test $\hat{P}$ against some other data.

Test traces $\quad x_{test}[0], \dots, x_{test}[T'] \quad$ } Collected from the real world.

$\qquad\qquad\quad u_{test}[0], \dots, u_{test}[T'-1]$

$\boxed{\text{Not used to estimate } P}$

Try predicting the next state

$$\hat{x}_{test}[t] = \hat{A}\,\vec{x}_{test}[t-1] + \hat{B}\,u_{test}[t-1]$$

learned parameters.

Test data

Compute Avg Test Error $= \frac{1}{T'} \sum_{t=1}^{T'} \left\| \hat{x}_{test}[t] - \vec{x}_{test}[t] \right\|^2$

Used to compute $\hat{\beta}$

Compare it to Training Error $= \frac{1}{T} \underbrace{\sum_{t=1}^{T} \left\| \hat{x}[t] - \vec{x}[t] \right\|^2}_{\text{Least Squares minimized this}}$

If close, and acceptable for the application, Good!

If Test error is significantly worse, bad.

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State: What's relevant about the past to predict the future.

Might not know state.

Guess: $\vec{x}[t]$

Another Guess $\begin{bmatrix} \vec{x}[t-1] \\ \vec{x}[t] \end{bmatrix}$

Yet Another Guess $\begin{bmatrix} \vec{x}[t-2] \\ \vec{x}[t-1] \\ \vec{x}[t] \end{bmatrix}$

Try all of these.

And choose the best one.

best on test data.

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We have a model. What now?

Divide our goal into 2 parts:

(1) Planning: Choose inputs $u[t]$ to achieve our goal assuming model is perfectly correct & nature cooperates.

(2) "Feedback Control": Responsive execution. Achieve goal despite disturbances

## Key Concept: Stability

$$x[t+1] = \lambda x[t] + u[t] + w[t]$$

$\lambda$ ← known; $u[t]$ ← Going to be our choice; $w[t]$ ← nature's choice.

Assume $u[t] = 0$ for now.

Are bad things going to happen because of $w[t]$?

$$x[t+1] = \lambda x[t] + w[t]$$

Basic question: Do things "blow up"

Simple example: $x[t+1] = 2x[t]$ Does this blow up?

If $x[0] = 0$, then we're fine.

$x[0] = 10^{-9}$

$x[32] = 2^{32} \cdot 10^{-9} \approx 4$ We don't like growing exponentials

$x[40] \approx 1000$

$x[t+1] = \frac{1}{2} x[t]$ Not blowing up.

First Attempt to define stability.

"Stay in the box" stability:

A system is stable iff $\exists K$ s.t. $\|x[t]\| \leq K \ \forall t \geq 0$

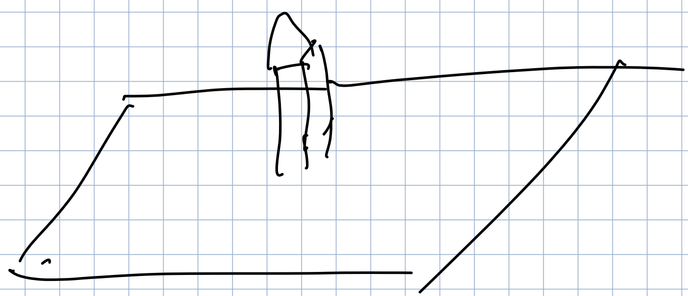
($\exists$: There exists; $\forall$: For all)

Clearly $K$ needs to depend on initial condition.

$x[t+1] = 2x[t] + w[t] \longrightarrow$ Unstable

$x[t+1] = \frac{1}{2} x[t] + w[t]$

Stability seems to depend on $w[t]$



Need to bound $w[t]$

New Definition: $\underbrace{\text{Bounded-input Bounded-output}}_{\text{BIBO}}$ stability

"$x[t]$ stays in a box as long as $|w[t]|$ is not too big"

The system is BIBO-stable iff $\forall \epsilon \geq 0$ $\exists K$ s.t.

if $\|w[t]\| \leq \epsilon$ $\forall t \geq 0$, then $\|x[t]\| \leq K$ $\forall t \geq 0$.

& $\|x[0]\| \leq \epsilon$

$$x[t+1] = \lambda x[t] + w[t]$$

Case 1 $|\lambda| > 1$. Not stable

Consider $x[0] = 0$, $w[0] = \epsilon$, $w[t>0] = 0$

$x[t] = ?$

$$x[0] = 0$$
$$x[1] = \epsilon$$
$$x[2] = \lambda \epsilon$$
$$\vdots$$
$$x[t] = \lambda^{t-1} \epsilon$$
$$|x[t]| = |\lambda^{t-1}| \epsilon$$
$$= |\lambda|^{t-1} \epsilon \xrightarrow{t \to \infty} \infty \text{ since } |\lambda| > 1$$

Case 2: $|\lambda| < 1$ $\Leftarrow$ Guess stable

Recall Discussion 2A or 6A:

$$x[t] = \lambda^t x[0] + \sum_{k=0}^{t-1} \lambda^k w[t-1-k]$$

Want $|x[t]| < K$

Consider $|x[t]| = \left|\lambda^t x[0] + \sum_{k=0}^{t-1} \lambda^k w[t-1-k]\right|$

I remember triangle inequality $|A+B| \leq |A| + |B|$

$$|x[t]| \leq |\lambda^t||x[0]| + \sum_{k=0}^{t-1} |\lambda^k||w[t-1-k]|$$

I know $|x[0]| \leq \epsilon$ $|w[\cdot]| \leq \epsilon$

$$|x[t]| \leq \epsilon \cdot |\lambda|^t + \epsilon \sum_{k=0}^{t-1} |\lambda|^k$$

$$< \epsilon \sum_{k=0}^{\infty} |\lambda|^k = \frac{\epsilon}{1 - |\lambda|} < \infty$$

So stable!

One case left $|\lambda| = 1$ $\lambda = e^{j\theta}$ for some real $\theta$