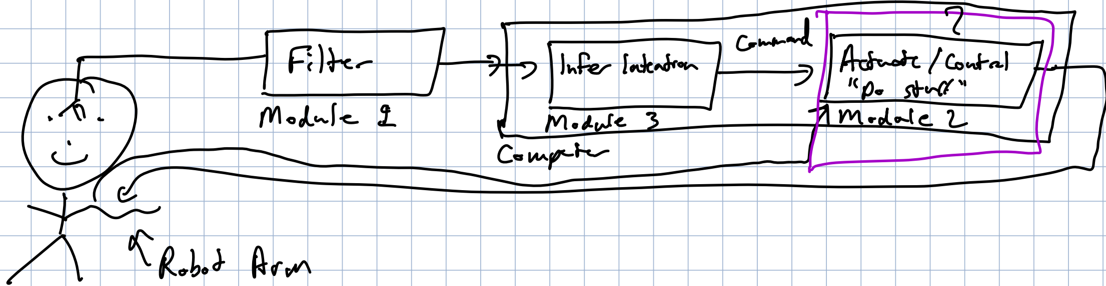
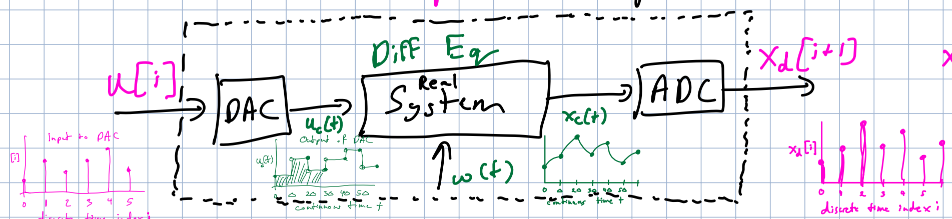
16B Prof. Anant Sahai

Reminder: In-Person Midterm Mon Oct 18 7–9 pm
Locations: TBA
Scope: Through HW 7 due Friday Oct 15th

Today: Finish stability basics
Feedback Control



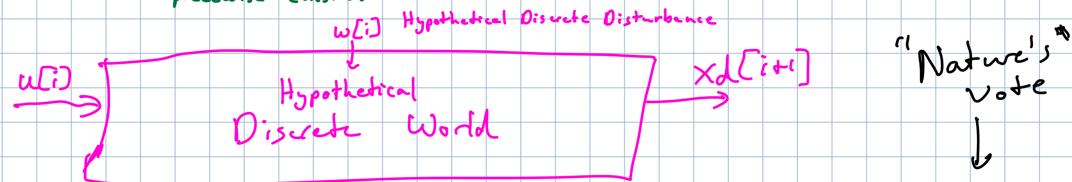
How does a computer experience the world?



$x_d[i] = x_c(i\Delta)$

$u_c(t) = u[i]$ if $t \in (i\Delta, (i+1)\Delta)$

piecewise-constant



Discrete model: $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i]$

Story so far: (1) Can use data to learn the system model using least squares.
(2) To achieve our goals using this model, we need to:

Defer this ⟶ (A) Make a plan for controls $\vec{u}[i]$ that achieve our goal

Present focus ⟶ (B) Understand how to reliably execute that plan interactively in the face of disturbances.

To do (B), we need to achieve stability:

The system is BIBO-stable iff $\forall \epsilon \geq 0 \ \exists K$ s.t.

if $\|\vec{w}[i]\| \leq \epsilon \ \forall i \geq 0$ & $\|\vec{x}[0]\| \leq \epsilon$, then $\|\vec{x}[i]\| \leq K \ \forall i \geq 0$

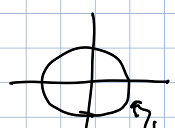
This way, the combined impact of the disturbances will be bounded on us.

Last time: considered $x[i+1] = \lambda x[i] + w[i]$

Saw: $|\lambda| > 1 \implies$ unstable : state can grow unboundedly

$|\lambda| < 1 \implies$ stable : $|x[i]| < \left(\frac{1}{1-|\lambda|}\right)\epsilon$

One case left: $|\lambda| = 1$. ie. $\lambda = e^{j\theta}$ for some real θ



unit circle

$$x[i+1] = e^{j\theta} x[i] + w[i]$$

Unroll:

$$x[i] = e^{j\theta i} x[0] + \sum_{k=0}^{i-1} e^{j\theta k} w[i-1-k]$$

Try $\theta = 0$ so $e^{j \cdot 0 \cdot i} = 1 \implies$ can blow up by picking $w[i] = \epsilon$

Let's try to get $\sum_{k=0}^{i-1} e^{j\theta k} w[i-1-k]$ to blow up as i increases.

Trying to prove $|\lambda| = 1$ implies unstable.

Want $w[i-1-k] \underset{\text{proportional}}{\approx} e^{-j\theta k}$

Try $w[i] = \epsilon e^{j\theta i}$

$$x[i] = e^{j\theta i} x[0] + \sum_{k=0}^{i-1} e^{j\theta k} \cdot \epsilon e^{j\theta(i-1-k)}$$

$$= e^{j\theta i} x[0] + \epsilon e^{j\theta(i-1)} \sum_{k=0}^{i-1} 1$$

$\to \infty$ as $i \to \infty$

So unstable.

Two directions: 1) Need to understand BIBO-stability for continuous time.

2) Vector systems.

$$\frac{d}{dt} x(t) = \lambda x(t) + w(t) \implies x(t) = \underline{e^{\lambda t} x(0)} + \int_0^t e^{\lambda(t-\tau)} w(\tau)\, d\tau$$

We know that if $Re(\lambda) > 0$, then

$$e^{\lambda t} = \underbrace{e^{Re(\lambda) \cdot t}}_{} \cdot e^{j\, Im(\lambda) t}$$

Blows up if $Re(\lambda) > 0$

So $Re(\lambda) > 0$ means unstable.

$Re(\lambda) < 0$ then initial condition doesn't make it blow up

Prove stability.

+

$$|x(t)| = \left| e^{\lambda t} x(0) + \int_0^t e^{\lambda(t-\tau)} w(\tau)\, d\tau \right|$$

$$\leq |e^{\lambda t}||x(0)| + \int_0^t |e^{\lambda(t-\tau)}| \cdot |w(\tau)|\, d\tau$$

$$= \underbrace{|e^{\lambda t}||x(0)|}_{\text{Not a problem}} + \int_0^t e^{Re(\lambda)(t-\tau)} \cdot |w(\tau)|\, d\tau$$

$$\leq |e^{\lambda t}| \epsilon + \epsilon \int_0^t e^{Re(\lambda)(t-\tau)}\, d\tau$$

$$\overset{\text{by calculus}}{=} |e^{\lambda t}| \epsilon + \epsilon \frac{\left(1 - e^{+Re(\lambda) t}\right)}{|Re(\lambda)|} < \frac{\epsilon}{|Re(\lambda)|}$$

<u>Summary</u>

	Discrete time	Continuous time
Stable	$\lvert\lambda\rvert < 1$	$Re(\lambda) < 0$
Unstable	$\lvert\lambda\rvert \geq 1$	$Re(\lambda) \geq 0$

<u>From scalar to vector</u> $\quad \vec{x}[i+1] = A\vec{x}[i] + \vec{w}[i]$

Conjecture: If all the eigenvalues of A have $|\lambda| < 1$, then BIBO-stable.

Why? Because we think $\exists$ coordinate system in which A might be "nice"

Need to show: If $\exists$ eigenvalue λ s.t. $|\lambda| \geq 1$, then unstable.

And if $\forall$ eigenvalue $|\lambda| < 1$, then stable.

Try $\vec{w}[i] = w_s[i] \cdot \vec{v}$ where $A\vec{v} = \lambda \vec{v}$

Consider $x_s[i+1] = \lambda x_s[i] + w_s[i]$ Choose $w_s[i]$ to make $|x_s[i]|$ blow up.

$$\vec{x}[i] = A^i \vec{x}[0] + \sum_{k=0}^{i-1} A^{i-1-k} \vec{w}[k]$$

$$= \underline{\quad\quad} + \sum_{k=0}^{i-1} A^{i-1-k} w_s[i] \vec{v}$$

$$= \underline{\quad\quad} + \sum_{k=0}^{i-1} \left(A^{i-1-k} \vec{v}\right) w_s[i]$$

$$= \underline{\quad\quad} + \vec{v} \underbrace{\sum_{k=0}^{i-1} \left(\lambda^{i-1-k} w_s[i]\right)}_{\text{Blows up}}$$

For $|\lambda| < 1$ for each eigenvalue can use diagonalization.

Assume A diagonalizable. $\exists V$ invertible s.t. $AV = V \Lambda$

Λ = diagonal$(\lambda_1, \dots, \lambda_n)$

Change coordinates to $\tilde{\vec{x}} = V^{-1} \vec{x}$

$$V^{-1} A V = \Lambda$$

$$V^{-1} \vec{x}[i+1] = V^{-1} A \vec{x}[i] + V^{-1} \vec{w}[i]$$

$$\tilde{\vec{x}}[i+1] = V^{-1} A V \tilde{\vec{x}}[i] + V^{-1} \vec{w}[i]$$

$$\tilde{\vec{x}}[i+1] = \Lambda \tilde{\vec{x}}[i] + \underbrace{V^{-1} \vec{w}[i]}$$

Is this bounded is $\|\vec{w}[i]\| \le \epsilon$?

$$V^{-1} = n \overset{n}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ & \ddots & & \cdot \end{bmatrix}}$$

largest entry has magnitude m

So $V^{-1} \vec{w}$ has each entry bounded by $n \cdot m \cdot \epsilon$

largest magnitude in V^{-1}

size of vectors n-dim

original bound

$$\tilde{x}_r[i+1] = \lambda_r \tilde{x}_r[i] + \underbrace{\left(V^{-1} \vec{w}[i]\right)_r}_{\text{bounded}}$$

$|\lambda_r| < 1$

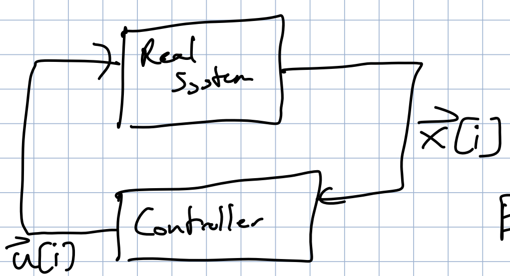
stable stays bounded.

Know conditions for stability. Nice eigenvalues of A.

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i]$$

What if A is unstable?

Can use feedback control.



How should we compute $\vec{u}[i]$ based on $\vec{x}[i]$?

Easiest : $\vec{u}[i] = \underbrace{F}_{\text{our choice}} \vec{x}[i]$

$$\vec{x}[i+1] = A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i]$$
$$= \underbrace{(A + BF)}_{\text{Can we make this be "nice"??}} \vec{x}[i] + \vec{w}[i]$$

First try scalar case: $x[i+1] = 2x[i] + u[i] + w[i]$

$2 \longrightarrow (2+f)$ if I use $u[i] = f \cdot x[i]$

$$x[i+1] = 2x[i] + f x[i] + w[i]$$
$$= (2+f)x[i] + w[i].$$

Can set $2+f$ to be anything I want!!

Can I get $(A+BF)$ to be any matrix that I want?

If B were invertible, then yes I can make $A+BF$ = anything

$$A + BF = G \implies BF = G - A$$
$$F = B^{-1}(G-A)$$

What if B was just a vector?

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] + \vec{w}[i]$$

$$u[i] = F\vec{x}[i] = [f_1 \quad f_2]\,\vec{x}[i]$$

$$A+BF = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[f_1 \quad f_2]$$

$$= \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 3+f_1 & 2+f_2 \end{bmatrix}$$

Can I get eigenvalues that I want?

$$\det\left(\lambda I - (A+BF)\right) = \det\begin{bmatrix} \lambda & -1 \\ -3-f_1 & \lambda-2-f_2 \end{bmatrix}$$

$$= \lambda(\lambda-2-f_2) - (3+f_1)$$

$$= \lambda^2 - (2+f_2)\lambda - (3+f_1)$$

Want λ_1, λ_2. $\Rightarrow$ Want characteristic polynomial to be

$$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$$

<u>Match coefficients!</u>

$$2+f_2 = \lambda_1+\lambda_2$$

$$-3-f_1 = \lambda_1\lambda_2$$

Can solve for f_1, f_2. ☺

Eigenvalue placement is possible, at least for this example.