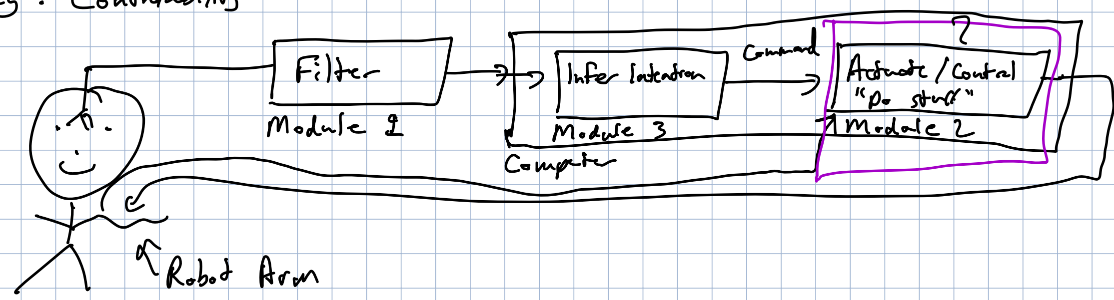
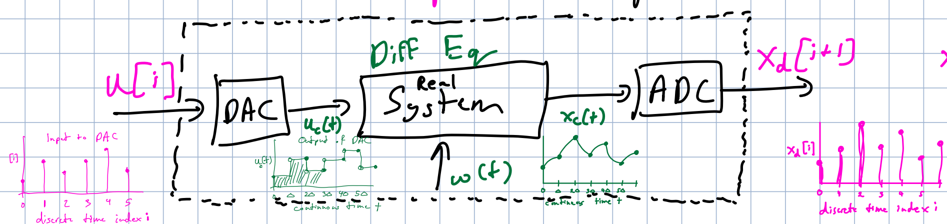
16B Prof Sahai

Midterm Oct 18 7-9 pm

Today: Controllability



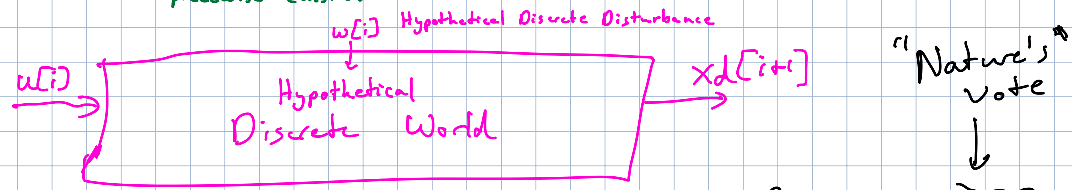
How does a computer experience the world?



$x_d[i] = x_c(i\Delta)$

$u_c(t) = u[i]$ if $t \in (i\Delta, (i+1)\Delta)$

piecewise-constant



Discrete model: $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i]$

Story so far: (1) Can use data to learn the system model using least squares

(2) To achieve our goals using this model, we need to:

Defer this ⟶ (A) Make a plan for controls $\vec{u}[i]$ that achieve our goals

Present focus ⟶ (B) Understand how to reliably execute that plan interactively in the face of disturbances.

To do (B), we need to achieve stability:

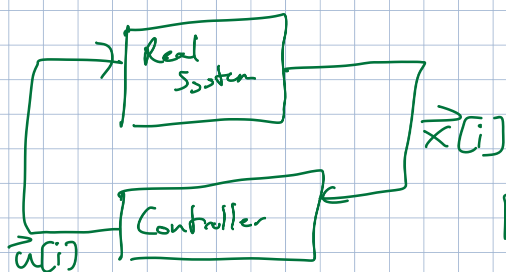
The system is BIBO-stable iff $\forall \epsilon \geq 0\ \exists K$ s.t.

if $\|\vec{w}[i]\| \leq \epsilon \ \forall i \geq 0$, then $\|\vec{x}[i]\| \leq K \ \forall i \geq 0$

& $\|\vec{x}[0]\| \leq \epsilon$

This way, the combined impact of the disturbances will be bounded on us

Can use feedback control.



How should we compute $\vec{u}[i]$ based on $\vec{x}[i]$?

Easiest: $\vec{u}[i] = F\,\vec{x}[i]$

our choice of control law

$$\vec{x}[i+1] = A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i]$$
$$= \underbrace{(A+BF)}_{A_{cl}}\,\vec{x}[i] + \vec{w}[i]$$

closed loop dynamics → A_{cl} Can we make this be "nice"??

Example: Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A_{cl} = A + BF = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 2+f_1 & f_2 \\ 0 & 3 \end{bmatrix}$$

$$\det(\lambda I - A_{cl}) = \det\begin{bmatrix} \lambda - 2 - f_1 & -f_2 \\ 0 & \lambda - 3 \end{bmatrix} = (\lambda-3)(\lambda-2-f_1)$$

$\lambda_1 = 3 \quad \lambda_2 = 2 + f_1$

Some cases aren't controllable. :(Why?

In this case $\vec{b}$ is an eigenvector of A: can't ifluence the other dimension at all.

Zoom in Focus on $n=2$ A is a 2×2 matrix.

Suspect: eigenvectorness of $\vec{b}$ is the problem.

How can we check if our suspicion is true???

Consider a general 2×2 $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ← specific case

What does it mean for $\vec{b}$ to not be an eigenvector?

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} \neq \lambda\vec{b} = \lambda\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \forall \lambda.$$

γ must be nonzero if $\vec{b}$ is not an eigenvector.

Is $\gamma \neq 0$ enough to let us place eigenvalues? Just try

$$A_{cl} = A + \vec{b}\,[f_1\ f_2] = \begin{bmatrix} \alpha + f_1 & \beta + f_2 \\ \gamma & \delta \end{bmatrix}$$

$$\det(A_{cl} - \lambda I) = \det \begin{bmatrix} \alpha + f_1 - \lambda & \beta + f_2 \\ \gamma & \delta - \lambda \end{bmatrix}$$

$$= (\delta - \lambda)(\alpha + f_1 - \lambda) - \gamma(\beta + f_2)$$

$=$ Algebra

$$= \lambda^2 - \lambda(\delta + \alpha + f_1) + (\delta\alpha + \delta f_1 - \gamma\beta - \gamma f_2)$$

If I want $\lambda^2 + c_1\lambda + c_0$

Then matching terms:

$$c_1 = -(\delta + \alpha + f_1) \Longrightarrow \boxed{f_1 = -c_1 - \delta - \alpha}$$

$$c_0 = \underbrace{\delta\alpha + \delta f_1 - \gamma\beta}_{\text{I know these}} - \gamma f_2$$

$\gamma \neq 0$

Solving: $$\boxed{f_2 = \frac{\delta\alpha - \delta(\delta + \alpha + c_1) - \gamma\beta - c_0}{\gamma}}$$

Wow! This worked if $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ What if $\vec{b} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Question: Can we change basis so $\tilde{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Want a basis $V = [\vec{v}_1\ \vec{v}_2]$ so $\vec{b} \longrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Set $\vec{v}_1 = \vec{b}$ What do we want for $\vec{v}_2$?

Need $\vec{v}_2 \neq \lambda \cdot \vec{b} \ \forall \lambda$
for V to be basis.

All we know is $\vec{b}$ is not an eigenvector of A.

Try $A\vec{b} \neq \lambda \cdot \vec{b} \ \forall \lambda$ So choose $\vec{v}_2 = A\vec{b}$

$$V = [\vec{b}\ \ A\vec{b}]$$

With this choice of coordinates what is the new dynamics $\tilde{A}$

$$\tilde{x} = V^{-1} x$$

Standard Diagram
For Coordinate Changes →

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \circ \arrow[d, bend right, "V^{-1}"'] \\
\tilde{x} \arrow[r, "\tilde{A} = V^{-1} A V"'] \arrow[u, bend right, "V"'] & \circ \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$\tilde{x}[i+1] = \tilde{A}\,\tilde{x}[i] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{V^{-1}\vec{b}} u[i] + V^{-1}\vec{w}[i]$$

$$\tilde{A} = V^{-1} A V = V^{-1}\left[A\vec{b} \quad A^2\vec{b}\right]$$

$$= \begin{bmatrix} 0 & \tilde{\beta} \\ 1 & \tilde{\delta} \end{bmatrix}$$

$\neq 0$ Woohoo!! That's what we needed.

So we've proved in the 2x2 case that $\vec{b}$ not being an eigenvector is enough to control with feedback.

Alternatively: If $\underbrace{[\vec{b}, A\vec{b}]}_{\text{Fact used above}}$ is full rank, then can control with feedback.

Q: If we can choose feedback in a nice coordinate system how do we choose it in our original one?

$$u[i] = [\tilde{f}_1 \ \tilde{f}_2]\,\tilde{x}[i] = \underbrace{[\tilde{f}_1 \ \tilde{f}_2]\, V^{-1}}_{\text{row. matrix}}\,\vec{x}[i]$$

row
$[f_1 \ f_2]$ ← in original coordinates.

If $[f_1 \ f_2] = [\tilde{f}_1 \ \tilde{f}_2] V^{-1}$

$\Rightarrow [\tilde{f}_1 \ \tilde{f}_2] = [f_1 \ f_2] V$

$$\tilde{A}_{cl} = \tilde{A} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[\tilde{f}_1 \ \tilde{f}_2]$$

$$= V^{-1} A V + V^{-1}\vec{b}\,[\tilde{f}_1 \ \tilde{f}_2]$$

$$= V^{-1} A V + V^{-1}\vec{b}\,[f_1 \ f_2]\, V = V^{-1}\left(A + \vec{b}\,[f_1 \ f_2]\right) V$$

Generalize beyond 2x2 case.

It turns out that $[\vec{b}, A\vec{b}]$ is the style that generalizes:

So for nxn case, look at $[\vec{b}, A\vec{b}, A^2\vec{b}, \dots, A^{n-1}\vec{b}]$
and see if full rank.

Goal: Show that rank $(B, AB, A^2B, \dots, A^{n-1}B) = n$
is a reasonable test for controllability.

what do we need? Need to be able to move the state.

$$\vec{x}[i+1] = A\,x[i] + B\vec{u}[i]$$ ⟵ ignore disturbance for now.

Can unroll:

$$\vec{x}[i] = A^i x[0] + \sum_{k=0}^{i-1} A^{i-1-k} B\vec{u}[k]$$

linear combinations of the columns of ?   ↖ our choices

Can see $\mathrm{span}\{B, AB, \dots, A^{i-1}B\}$ tells us where we can set $\vec{x}[i]$ to go.

If this span were n-dimensional, we could go wherever we want.

Definition. A system is controllable if given any target goal $\vec{g}$ we can find a time ℓ s.t. $\exists\ \vec{u}[0], \vec{u}[1], \dots, \vec{u}[\ell-1]$ so that $\vec{x}[\ell] = \vec{g}$.

How can I check for controllability? First attempt....

Start with $\ell = 1$ $C = B$

if span$[C]$ is n-dimensional, return controllable.

otherwise increment ℓ

Set $C = \{B, AB, \dots, A^{\ell-1}B\}$

Goto

Try $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Is $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ 2-dimensional? No

Is $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}$ 2-dimensional? No

Is $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}\right\}$ 2-dim? No

$\vdots$

How to stop? Oh no! Infinite loop.

Guess a property: Once span stops growing, it won't grow again

~~Need to state property~~

Thm: If $A^k \vec{b}$ is linearly dependent on $\{\vec{b}, A\vec{b}, \dots, A^{k-1}\vec{b}\}$

then $A^{k+1}\vec{b}$ is also linearly dependent on

Next time: Prove this...