16B Prof. Anant Sahai

Due dates extended for HW 8 & MT1 REDOs

Due to release on Mon instead of Sat. +2 days

Today: Upper-triangularization

Last time: Introduced G-S orthonormalization.

Loose end: How can we deal with "A" $n \times n$ matrices that do not have n distinct eigenvectors.

$$\begin{array}{ccc} \vec{x} & \xrightarrow{A} & \\ U^{-1}\uparrow\downarrow U & & U^{-1}\uparrow\downarrow U \\ \tilde{x} & \longrightarrow & \end{array}$$

$\tilde{A} = U^{-1} A U$

↑ Goal: "nice"

Previously, if A had n distinct eigenvectors, we could choose $U = [\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n]$ and the $\tilde{A}$ would be diagonal.

$\Longrightarrow$ Would turn a vector problem into n independent scalar problems.

Not always possible to find n distinct eigenvectors. e.g. RLC critically damped.

Recall HW 4 → $n=2$ but only one eigenvector $A\vec{v}_1 = \lambda \vec{v}_1$

There we used $\vec{v}_1$, and then we just "YOLO"-ed it with a second vector.

We got $\tilde{A} = \begin{bmatrix} \lambda & * \\ 0 & \lambda \end{bmatrix}$ (Something $\neq 0$)

Not Diagonal

Still nice though

We got a scalar problem and then another scalar problem that had the first scalar problem as an input.

Upper-triangular. ← has 0's below the diagonal.

We got chained scalar problems.

Also nice enough.

Can we always find a basis U s.t.

$$\tilde{A} = U^{-1} A U = \begin{bmatrix} \lambda_1 & * & * & * \\ & \lambda_2 & * & * \\ & & \lambda_3 & * \\ 0 & & \ddots & * \\ & & & \lambda_n \end{bmatrix}$$

"stuff" ← Goal

e.g.

$$\begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Need $U = [\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n]$

Start with 2×2 matrix case. $n=2$.

Picking $\vec{u}_1$ to be an eigenvector of A. So $A\vec{u}_1 = \lambda_1 \vec{u}_1$.

Pick $\vec{u}_1$ to have $\|\vec{u}_1\| = 1$.

Can do this because all matrices have at least one eigenvector.

In the HW, we saw that any choice for the second vector would have worked as long as it was linearly indep of $\vec{u}_1$.

So pick $\vec{r}_1$ as our second vector so $\|\vec{r}_1\| = 1$ & $\vec{r}_1 \perp \vec{u}_1$

i.e. $\vec{r}_1^T \vec{u}_1 = 0$

Our basis $U = [\vec{u}_1, \vec{r}_1]$ $\quad U^{-1} = U^T = \begin{bmatrix} \vec{u}_1^T \\ \vec{r}_1^T \end{bmatrix}$

by orthonormality of U

$$\tilde{A} = U^{-1} A U = \begin{bmatrix} \vec{u}_1^T \\ \vec{r}_1^T \end{bmatrix} A \begin{bmatrix} \vec{u}_1 & \vec{r}_1 \end{bmatrix} = \begin{bmatrix} \vec{u}_1^T \\ \vec{r}_1^T \end{bmatrix} \begin{bmatrix} A\vec{u}_1 & A\vec{r}_1 \end{bmatrix}$$

Easier to work with transpose instead of inverse

$$= \begin{bmatrix} \vec{u}_1^T \\ \vec{r}_1^T \end{bmatrix} \begin{bmatrix} \lambda_1\vec{u}_1 & A\vec{r}_1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \underbrace{\vec{u}_1^T\vec{u}_1}_{1} & \vec{u}_1^T A \vec{r}_1 \\ \lambda_1 \underbrace{\vec{r}_1^T\vec{u}_1}_{0} & \vec{r}_1^T A \vec{r}_1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix} \leftarrow \text{upper triangular}$$

Question: How can I find $\vec{r}_1$?

General Question: Given a vector $\vec{v}$, $\|\vec{v}\| = 1$, can I get an orthonormal basis that has $\vec{v}$ as its first element.

Want Orthonormal $[\vec{v}, \vec{q}_2, \vec{q}_3, \ldots \vec{q}_n]$.

Trick: Run GS on $[\vec{v}, \underbrace{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n}_{n+1 \text{ vectors.}}]$ where $\vec{e}_i$ is the i-th column of identity matrix.

Last lecture we implicitly assumed eigvecs was real.

Challenge: this set is definitely linearly dep.

Just run G.S. It will generate $\vec{z}_i$. (What residuals you get after projections.)

But one of these $\vec{z}_i$ will be a $\vec{0}$.

Just throw that one out.

At the end, we'll have n vectors that will span the whole space

Moving beyond 2×2.

Try 3×3 case. A

Pick $\vec{v}_1$ s.t. $A\vec{v}_1 = \lambda_1 \vec{v}_1$ & $\|\vec{v}_1\| = 1$ ← Works for general n.

Using the above G.S. procedure, we can get $\vec{r}_1, \vec{r}_2$

Call this $R = [\vec{r}_1, \vec{r}_2]$ ← Works for general n $[\vec{r}_1, \vec{r}_2, \dots, \vec{r}_{n-1}]$

s.t. $[\vec{v}_1 \;\; R]$ is orthonormal. $[\vec{v}, R]^T [v_1 \;\; R] = I$

Use basis $\underset{||}{V}$ to change coordinates $\begin{bmatrix} \vec{v}_1^T \\ \vec{r}_1^T \\ \vec{r}_2^T \end{bmatrix} = \begin{bmatrix} v_1^T \\ R^T \end{bmatrix} [v_1 \;\; R] = I$

$$\begin{bmatrix} v_1^T v_1 & v_1^T R \\ R^T v_1 & R^T R \end{bmatrix} = I$$

scalar $= 1$ since $\|\vec{v}_1\| = 1$

$v_1^T R \to \overbrace{[0 \;\; 0]}^{2}\}1$

$R^T v_1$: $2\{\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{1}$ ← Since $\vec{r}_1 \perp \vec{v}_1$, $\vec{r}_2 \perp \vec{v}_1$

$R^T R$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, since R is orthonormal.

$$\begin{aligned}
V^{-1} A V &= V^T A V \\
&= \begin{bmatrix} \vec{v}_1^T \\ R^T \end{bmatrix} A \begin{bmatrix} \vec{v}_1 & R \end{bmatrix} \\
&= \begin{bmatrix} \vec{v}_1^T \\ R^T \end{bmatrix} \begin{bmatrix} A\vec{v}_1 & AR \end{bmatrix} \\
&= \begin{bmatrix} \vec{v}_1^T \\ R^T \end{bmatrix} \begin{bmatrix} \lambda \vec{v}_1 & AR \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 \vec{v}_1^T \vec{v}_1 & \vec{v}_1^T AR \\ \lambda R^T \vec{v}_1 & R^T AR \end{bmatrix} \\
&= \begin{bmatrix} \lambda & \vec{v}_1^T AR \\ \vec{0} & \boxed{R^T AR} \end{bmatrix}
\end{aligned}$$

Works for general n

Note: For expository convenience, we are going to assume that all eigenvalues, eigenvectors we find are real.

↑ Progress towards upper-triangular.

This is a 2x2 matrix. $\underset{n-1}{\overset{n-1}{\square}} = \underset{n-1}{\overset{n}{\boxed{R^T}}}\;\underset{n}{\overset{n}{\boxed{A}}}\;\underset{n}{\overset{n-1}{\boxed{R}}}$

Is this upper-triangular! Who knows?!!?

☹ Are we stuck!?

Recurse! (And hope ...)

We know $\exists\, U_2$ s.t. U_2 is orthonormal (so $U_2^T = U_2^{-1}$)

↑ 2x2

and $U_2^T (R^T A R) U_2 = T_2 = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ ← upper-triangular

$= (U_2^T R^T) A (R U_2)$

$$= (R\,U_2)^T A (R\,U_2) = T_2$$

2x2 upper-triangular matrix.

Our goal is to have an orthonormal basis $U = [\vec{v}_1 \; - \; -]$
So that $U^T A U$ is upper-triangular.

Suggests using $R\,U_2$ instead of R in original basis.

So if I used $U = [\vec{v}_1 \;\; R\,U_2]$

Then $U^T A U = \begin{bmatrix} \lambda & \vec{v}_1^T A R U_2 \\ \vec{0} & (RU_2)^T A (RU_2) \end{bmatrix}$ Hope

$$= \begin{bmatrix} \lambda & \text{---} \\ \vec{0} & T_2 \end{bmatrix}$$ upper-triangular

Hope hinges on $[v_1 \; (R\,U_2)]$ being orthonormal

Are $(R\,U_2)$ $\perp$ to $\vec{v}_1$? ✓ So we get

$$\vec{v}_1^T R\,U_2 = (\vec{v}_1^T R)\,U_2 = \vec{0}^T U_2 = \vec{0}^T$$

Is $(R\,U_2)^T R\,U_2 = I$?

$$U_2^T R^T R\,U_2 = U_2^T I\,U_2 = U_2^T U_2 = I$$ ✓

Since $(AB)^T = B^T A^T$

We got an orthonormal basis U in which A is U.T.

Can we do this in general? When did we use "2"? or 3?

We needed to be able to upper-triangularize an arbitrary $(n-1) \times (n-1)$ matrix using an orthonormal basis U_{n-1}

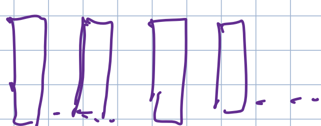
Then $U = [\vec{v}_1 \;\; R\,U_{n-1}]$ would work.

So we can do it by induction on n.

Assume it works for $(n-1)\times(n-1)$.
We just showed it worked for $n \times n$.

Need Base Case.

Induction Picture



Show if n^{th} domino falls, so does $n+1$

Show first domino falls.

$\Downarrow$

All dominos fall.

Base Case: ~~2x2~~ 1×1 $\quad [a] \quad U = [1]$

Already upper-triangular

How to Upper-Triangularize:

$UT(A)$: returns a pair U_n orthonormal & T upper-triangular s.t. $U^T A U = T$

n-dim

$UT(A)$:

if A is 1-dim, return $(U = [1],\ T = A)$

else. Let $\vec{v}_1$ be an eigenvector of A with eigenvalue λ.

If complex

Use $\boxed{G.S.}$ to construct $n-1$ vectors $R = [\vec{r}_1, \ldots, \vec{r}_{n-1}]$ s.t. $[\vec{v}_1 \;\; R]$ is orthonormal.

Compute $B = R^{\boxed{T}} A R$. $\longleftarrow$ $(n-1) \times (n-1)$ matrix.

Let $U', T' = UT(B)$

Set $U = [u_1 \;\; R U']$

$T = U^{\boxed{T}} A U$

Return (U, T)

Schur Decomposition — upper-triangularization.