16B. Prof. Anant Sahai

Today: Finish Uppertriangularization story
Understand consequences for stability
Symmetric Matrix Case

Announce: Remember Redo Self-Grades & HW 9 on regular schedule

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Recall: $\hat{A} = U^T A U$ where $U^T U = I$ since U is an orthonormal basis (U's columns)

Upper-triangular: $\hat{A} = \begin{bmatrix} \lambda_1 & \cdots & \cdots \\ & \lambda_2 & \cdots \\ 0 & & \lambda_3 & \cdots \\ & & & \lambda_n \end{bmatrix}$ "stuff"

Loose-end I: What are the eigenvalues of $\tilde{A}$?

a) Are these the same as the eigenvalues of A?

b) What are they? λ_i's.

λ_i is an eigenvalue of $\tilde{A}$

entry on the diagonal of $\hat{A}$.

Check Consider $\underbrace{\tilde{A} - \lambda I}_{\hat{A}_\lambda}$: $\begin{bmatrix} \lambda_1-\lambda & \cdots & \cdots \\ & \lambda_2-\lambda & \cdots \\ 0 & & \ddots \\ & & & \lambda_n-\lambda \end{bmatrix}$

We know that for an eigenvalue λ, this matrix is not invertible.
i.e. it has lin dep columns.

Also we want if $\lambda \neq \lambda_i$ for any i, then that λ is not an eigenvalue.

In this case, none of the diagonal entries are zero.

Claim: This implies $\tilde{A}_\lambda$ is invertible.

Why? Gaussian Elimination Succeeds! Can back-substitute to uniquely solve eqns $\tilde{A}_\lambda \vec{x} = \vec{b}$

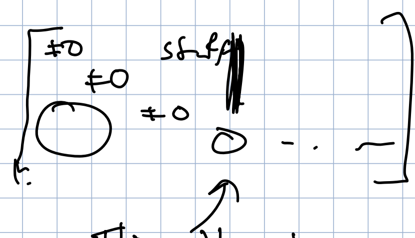
$\Downarrow$ Matrix can be inverted.

Other direction:
If $\lambda = \lambda_i$ for some i, then λ is an eigenvalue

i.e. $\tilde{A}_\lambda$ is not invertible if it has a 0 on the diagonal.

Why does having a zero on the diagonal imply linearly dep columns?

A: G.E. will get stuck. Won't find a pivot there.



Alternatively: This ith column is a linear combo of the colums before it.
Because we can solve for the weights by back substitution

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a) Are these λ_i the same eigenvalues as A?

Characteristic Poly

$$\det(\lambda I - \tilde{A}) = \det(\lambda I - U^T A U)$$
$$= \det(\lambda U^T U - U^T A U)$$
$$= \det(U^T(\lambda I - A)U)$$
$$= \det(U^T)\det(\lambda I - A)\det(U)$$
$$= \det(\lambda I - A)\det(U^T)\det(U)$$
$$= \det(\lambda I - A)\det(\cancel{U^T U})$$
$$= \det(\lambda I - A) \qquad \det(I) = 1$$

Same characteristic polys $\Longrightarrow$ same eigenvalues. ☺

So how does this help with BIBO stability?

Original System

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{w}[i] \qquad \|w[i]\| \le \epsilon$$

Claim: This is BIBO stable if all the eigenvalues of A have $|\lambda| < 1$

We already proved it when A was diagonalizable.

Want to show $\exists K$ s.t. $\|\vec{x}[i]\| \leq K \cdot \epsilon$.

By Upper-triangularization (Schur Decomposition)

$\exists U$ s.t. $\hat{A} = U^T A U$ is upper triangular.

$$\tilde{A} = \begin{bmatrix} \lambda_1 & & \beta_{i,j} \leftarrow \text{stuff} \\ & \lambda_2 & \\ 0 & \ddots & \lambda_{n-1} \quad \beta_{n-1,n} \\ & & \lambda_n \end{bmatrix}$$

Change coordinates to $\tilde{x} = U^T \vec{x}$

$$\begin{tikzcd} x \arrow[r, "A"] \arrow[d, shift right, "U^T"'] & \circ \arrow[d, shift right, "U^T"'] \\ \tilde{x} \arrow[r, "\tilde{A}"] \arrow[u, shift right, "U"'] & \circ \arrow[u, shift right, "U"'] \end{tikzcd}$$

$\tilde{A} = U^T A U$

$$\vec{\tilde{x}}[i+1] = \vec{\tilde{A}}\, \tilde{x}[i] + \underbrace{U^T \vec{\omega}[i]}$$

Is this bounded? Yes!

Call this $\vec{v}[i]$

$$\|U^T \vec{\omega}\|^2 = (U^T \vec{\omega})^T (U^T \vec{\omega}) \quad \text{By } \epsilon$$
$$= \vec{\omega}^T U U^T \vec{\omega}$$
$$= \vec{\omega}^T \vec{\omega} \quad \text{since } UU^T$$
$$= \|\vec{\omega}\|^2 \quad = U^T U = I$$

we know $\|\vec{v}[i]\| \leq \epsilon$

Use the philosophy of strong induction (Backwards)

Start at the end:

$$\boxed{\tilde{x}_n[i+1] = \lambda_n \tilde{x}[i] + v_n[i]}$$

Because $|\lambda_n| < 1$, This is BIBO stable.

$\exists$ constant K_n s.t. $|\tilde{x}_n| \leq K_n \epsilon$.

in particular $K_n = \dfrac{1}{1 - |\lambda_n|}$

Base case Done.

Consider $\tilde{x}_{n-1}$

$$\tilde{x}_{n-1}[i+1] = \lambda_{n-1}\,\tilde{x}_{n-1}[i] + \boxed{\beta_{n-1,n}\,\tilde{x}_n[i] + v_{n-1}[i]}$$

I know $|\lambda_{n-1}| < 1$

I want to say $\exists K_{n-1}$ s.t. $|\tilde{x}_{n-1}| \le K_{n-1}\epsilon$

To get what I want, I need to be bounded.

But $\left|\beta_{n-1,n}\,\tilde{x}_n[i] + v_{n-1}[i]\right| \le |\beta_{n-1,n}|\,|\tilde{x}_n[i]| + |v_{n-1}[i]|$

$$\le |\beta_{n-1,n}|\cdot K_n\epsilon + \epsilon$$

specific number

$$\le \left(|\beta_{n-1,n}|\cdot K_n + 1\right)\epsilon$$

$$K_{n-1} = \left(\frac{1}{1-|\lambda_{n-1}|}\right)\left(|\beta_{n-1,n}|\cdot K_n + 1\right)$$

Assume $\forall k$ in $l+1, l+2, \dots, n$,

we know $\exists K_k$ s.t. $|\tilde{x}_k[i]| \le K_k\cdot\epsilon$

Want to show $|\tilde{x}_l[i]| \le K_l\cdot\epsilon$.

$$\tilde{x}_l[i+1] = \lambda_l\,\tilde{x}_l[i] + \underbrace{\left(\sum_{k=l+1}^{n}\beta_{l,k}\,\tilde{x}_k[i]\right) + v_l[i]}$$

is bounded by $\underbrace{\left(\left(\sum_{k=l+1}^{n}|\beta_{l,k}|\,K_k\right)+1\right)\epsilon}_{\star}$

So $\exists K_l = \left(\frac{1}{1-|\lambda_l|}\right)\star$ s.t. $|\tilde{x}_l| \le K_l\cdot\epsilon$

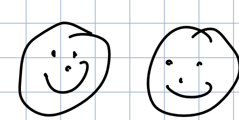
$$\tilde{A} = \begin{bmatrix} \lambda_1 & & \beta_{i,j} & \\ & \lambda_2 & & \\ & & \ddots & \\ & 0 & \lambda_{n-1} & \beta_{n-1,n} \\ & & & \lambda_n \end{bmatrix} \leftarrow \text{stuff} \quad = U^T A U$$

Recall $U^T U = I$

How do we know that these $\beta_{i,j}$ are well behaved?

Could these $\beta_{i,j}$ be "ginormous"? ← Unreasonably huge!

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$$

$\boxed{\text{NO}}$ 

Want to understand $U^T A U$

First let's look at $(U^T A) = \begin{bmatrix} U^T \vec{a}_1 & U^T \vec{a}_2 & \cdots & U^T \vec{a}_n \end{bmatrix}$

$$\| U^T \vec{a}_i \|^2 = \vec{a}_i^{\,T} U U^T \vec{a}_i = \vec{a}_i^{\,T} I \vec{a}_i$$
$$= \vec{a}_i^{\,T} \vec{a}_i = \| \vec{a}_i \|^2$$
$$= \sum_{j=1}^{n} a_{ji}^2$$

Fact: The sum of squares of entries in $U^T A$ is the same as the sum of squares of entries in A.

<u>Definition</u> The Frobenius Norm $\| A \|_F^2 = \sum_{i,j} (a_{ij})^2$

Just Proved: If U is orthonormal square matrix then

$$\| U A \|_F = \| A \|_F$$

From the definition: $\| A \|_F = \| A^T \|_F$

$$\| \tilde{A} \|_F = \| U^T A U \|_F = \| (U^T A U)^T \|_F$$

$$= \|U^T (A^T U)\|_F$$
$$= \|A^T U\|_F$$
$$= \|(U^T A)^T\|_F$$
$$= \|U^T A\|_F$$
$$= \|A\|_F$$

$$\|AU\|_F = \|U^T A^T\|_F = \|A^T\|_F = \|A\|_F$$

Since $U^T U = I$

$\Longrightarrow U U^T = I$ since U is square.

So $\tilde{A}$ has big entries only if A had them.

We know $\tilde{A} = U^T A U$ is upper-triangular.

Let's just play.

$$\tilde{A} = \begin{bmatrix} \lambda_1 & & \beta_{i,j} & \\ & \lambda_2 & & \\ 0 & & \lambda_{n-1} & \beta_{n-1,n} \\ & & & \lambda_n \end{bmatrix}$$ ← stuff

$$\tilde{A}^T = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ \beta_{j,i} & & \lambda_n \end{bmatrix}$$

Lower-triangular.

$$\tilde{A}^T = (U^T A U)^T = U^T A^T U$$

What if $A = A^T$ i.e. A was a symmetric matrix?

$$\Longrightarrow \tilde{A}^T = \tilde{A} \Longrightarrow \beta_{i,j} = 0 \quad \forall\, i,j$$

$\tilde{A}$ would be diagonal. Λ.

$$\tilde{A} = \Lambda = U^T A U$$

$$\Longrightarrow A = U \Lambda U^T$$

A would have to have a full complement of eigenvectors that are orthogonal to each other

Since $\vec{u}_i$ is an eigenvector!

$$A\vec{u}_i = U\Lambda U^T \vec{u}_i = U\Lambda \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow i\text{th position}$$

$$= U \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda_i \\ 0 \\ \vdots \end{bmatrix} = \lambda_i \vec{u}_i$$

Any real symmetric matrix $\boxed{\text{with all real eigenvalues}}$ has orthogonal eigenvectors and a full complement of them

Question: Must real symmetric matrices have real eigenvalues?

Will do this next time....