16 B Make-up lecture due to zoom recording glitch ☹

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Today: Finish Uppertriangularization story: eigenvalues
Understand consequences for stability
Symmetric Matrix Case

Recall: $\hat{A} = U^T A U$ where $U^T U = I$ since U has orthonormal columns that form a basis for n-dim

Upper-triangular:
$$\hat{A} = \begin{bmatrix} \lambda_1 & \cdots & \cdots & \cdots \\ & \lambda_2 & \cdots & \cdots \\ 0 & & \lambda_3 & \cdots \\ & & & \lambda_n \end{bmatrix}$$
"stuff"

From last lecture

How to Upper-Triangularize: ← Assuming all real eigenvalues.

$UT(A)$: returns a pair U orthonormal
n-dim
& T upper-triangular
s.t. $U^T A U = T$

$UT(A)$:

if A is 1-dim, return $(U = [1],\ T = A.)$

else. Let $\vec{v}_1$ be a unit eigenvector of A with eigenvalue λ.

Use $\boxed{\text{G.S.}}$ to construct $n-1$ vectors $R = [\vec{r}_1, \ldots, \vec{r}_{n-1}]$
s.t. $[\vec{v}_1, R]$ is orthonormal.

Compute $B = R^{\boxed{T}} A R$. ⟵ $(n-1) \times (n-1)$ matrix.

Let $U', T' = UT(B)$

Set $U = [\vec{v}_1, R U']$

$T = U^{\boxed{T}} A U$

Return (U, T)

If complex
These boxes would have to change

Schur Decomposition — upper-triangularization.

$\tilde{A} = U^T A U$ where $U^T U = I$ since U has n orthonormal column that form a basis for n-di

↑ Upper-triangular:

$$\tilde{A} = \begin{bmatrix} \lambda_1 & \cdots & \cdots & \cdots \\ & \lambda_2 & \cdots & \cdots \\ & & \lambda_3 & \vdots \\ 0 & & & \lambda_n \end{bmatrix}$$

"stuff"

2 Questions

1) What are the eigenvalues of $\tilde{A}$?
   ↳ The $\lambda_1, \lambda_2, \cdots, \lambda_n$ on the diagonal of $\tilde{A}$
2) How are the eigenvalues of $\tilde{A}$ related to the eigenvalues of A?
   ↳ They're the same.

1) General Fact: If a matrix is upper-triangular, the entries on the diagonal are the eigenvalues.

Consider
$$\tilde{A}_\lambda = (\tilde{A} - \lambda I)$$
We know that if λ is an eigenvalue of $\tilde{A}$, then the matrix $\tilde{A}_\lambda$ is not invertible. And vice-versa.

If $\lambda \neq \lambda_i$ for any i, then λ is not an eigenvalue of $\tilde{A}$.

If $\lambda \neq \lambda_i$, then
$$\tilde{A}_\lambda = \begin{bmatrix} \lambda_1 - \lambda & & \cdots & \vdots \\ & \lambda_2 - \lambda & \cdots & \vdots \\ 0 & & \ddots & \\ & & & \lambda_n - \lambda \end{bmatrix}$$
has all nonzero entries on the diagonal.

$\Longrightarrow$ $\tilde{A}_\lambda$ is invertible.

Since $\tilde{A}_\lambda \vec{x} = \vec{b}$ is uniquely solveable by back-substitution.

$\Longrightarrow$ λ is not an eigenvalue of $\tilde{A}$.

Other direction: If $\lambda = \lambda_i$ for some i, then λ is an eigenvalue of $\tilde{A}$.

Need to show that
$$\tilde{A}_\lambda = \begin{bmatrix} \lambda_1 - \lambda & & & & & & \\ & \ddots & & & & & \\ & & \lambda_{i-1}-\lambda & & & & \\ & & & 0 & & & \\ & & & & \lambda_{i+1}-\lambda & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n \end{bmatrix}$$

is not invertible. $\Longrightarrow$ Clearly Gaussian elimination will fail to find a pivot at i^{th} step.

Consider first i columns of $\hat{A}_\lambda$.

Claim. $\vec{\tilde{a}}_{\lambda,i}$ is a linear combination of the first $i-1$ columns.

$$\begin{array}{cc} & \\ & \\ & \\ & \\ & \\ i^{th}\text{ row} & \\ & \\ & \\ & \end{array}\begin{bmatrix} \lambda_1-\lambda & * & \cdots & * & * \\ 0 & \lambda_2-\lambda & \cdots & * & * \\ 0 & 0 & & * & * \\ 0 & 0 & & * & * \\ \vdots & \vdots & & \lambda_{i-1}-\lambda & * \\ \vdots & \vdots & & 0 & 0 \\ \vdots & \vdots & & 0 & 0 \\ & & & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \end{bmatrix}$$

$\uparrow$ i^{th} column of $\tilde{A}_\lambda$

Can use back substitution to solve for weights $\alpha_1, \ldots, \alpha_{i-1}$ s.t.
$$\vec{\tilde{a}}_{\lambda,i} = \sum_{j=1}^{i-1} \alpha_j \vec{\tilde{a}}_{\lambda,j}$$

$\Longrightarrow$ shows linear dependence.

$\Longrightarrow$ $\tilde{A}_\lambda$ has a nullspace that is nontrivial

$\Longrightarrow$ λ is an eigenvalue of $\tilde{A}$.

Claim Eigenvalues of $\tilde{A}$ are the same as eigenvalues of A.

Consider: $\det(\lambda I - \tilde{A}) = \det(\lambda I - U^T A U)$
$$= \det(\lambda U^T U - U^T A U)$$

$$= \det(U^T(\lambda I - A)U)$$
$$= \det(U^T)\det(\lambda I - A)\det(U)$$
$$= \det(\lambda I - A)\det(U^T)\det(U)$$
$$= \det(\lambda I - A)\det(U^T U)$$
$$= \det(\lambda I - A)\underbrace{\det(I)}_{1}$$
$$= \det(\lambda I - A)$$

The two characteristic polynomials are the same.

$\Longrightarrow$ eigenvalues are the same.