16 B Make-up lecture due to zoom recording glitch ☹

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Today: Finish Uppertriangularization story: eigenvalues
Understand consequences for stability
Symmetric Matrix Case

Recall: $\widehat{A} = U^T A U$ where $U^T U = I$ since U has orthonormal columns that form a basis for n-dim

Upper-triangular: $\widehat{A} = \begin{bmatrix} \lambda_1 & - & - & - & - \\ & \lambda_2 & - & - & - \\ & & \lambda_3 & - & - \\ & 0 & & \ddots & \vdots \\ & & & & \lambda_n \end{bmatrix}$ "stuff"

These are the eigenvalues of A.

BIBO Stability via Upper-triangularization

$\vec{x}[i+1] = A\vec{x}[i] + \vec{w}[i]$ know $\|\vec{w}[i]\| \le \epsilon$

n-dimensional

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, bend right, "U^T"'] & \bullet \arrow[d, bend right, "U^T"'] \\ \tilde{\vec{x}} \arrow[r, "\tilde{A}"', "U^T A U"] \arrow[u, bend right, "U"'] & \bullet \arrow[u, bend right, "U"'] \end{tikzcd}$$

Claim: This system is BIBO stable if all the eigenvalues of A have $|\lambda| < 1$.

Already proven this if A were diagonalizable. ⇐ A little unsatisfying because we had a V^{-1} showing up in our bounding.

We assume we can uppertriangularize by U.

$\tilde{\vec{x}}[i+1] = \tilde{A}\tilde{\vec{x}}[i] + \underbrace{U^T\vec{w}[i]}_{\text{Is this bounded?}}$

Yes since $\|U^T\vec{w}\|^2 = (U^T\vec{w})^T(U^T\vec{w})$
$= \vec{w}^T U U^T \vec{w}$

But $U^T = U^{-1}$ and so $UU^T = I$
$= \vec{w}^T\vec{w}$
$= \|\vec{w}\|^2$

Call $\vec{v}[i] = U^T\vec{w}[i]$

and we know $\|\vec{v}[i]\| \le \epsilon$

Use the philosophy of strong induction (done backwards)

$$\vec{\tilde{x}}[i+1] = \tilde{A}\,\vec{\tilde{x}}[i] + \underbrace{U^T \vec{\omega}[i]}_{\vec{\tilde{v}}[i]}$$

$$\begin{array}{l} \\ \\ \ell^{th}\text{ row} \\ \\ \\ \text{bounded} \\ \text{bounded} \end{array}
\begin{bmatrix} \tilde{x}_1[i+1] \\ \tilde{x}_2[i+1] \\ \vdots \\ \vdots \\ \tilde{x}_{n-2}[i+1] \\ \tilde{x}_{n-1}[i+1] \\ \tilde{x}_n[i+1] \end{bmatrix}
=
\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \beta_{i,j} & \\ & & \ddots & & \\ & O & & \lambda_{n-2} & \\ & & & \lambda_{n-1} & \beta_{n-1,n} \\ & & & & \lambda_n \end{bmatrix}
\begin{bmatrix} \tilde{x}_1[i] \\ \tilde{x}_2[i] \\ \vdots \\ \vdots \\ \tilde{x}_{n-1}[i] \\ \tilde{x}_n[i] \end{bmatrix}
+
\begin{bmatrix} v_1[i] \\ \vdots \\ \vdots \\ \vdots \\ v_{n-1}[i] \\ v_n[i] \end{bmatrix}$$

Start with the last row.

$$\tilde{x}_n[i+1] = \lambda_n \tilde{x}_n[i] + \underbrace{v_n[i]}_{\text{Bounded by } \epsilon}$$

Know $|\lambda_n| < 1$

$\Longrightarrow$ This is BIBO stable.

$\exists$ constant K_n s.t. $|\tilde{x}_n[i]| \leq K_n \cdot \epsilon$

In particular $K_n = \dfrac{1}{1 - |\lambda_n|}$

<u>Base case for induction is now done.</u>

Next row up.

$$\tilde{x}_{n-1}[i+1] = \lambda_{n-1} \tilde{x}_{n-1}[i] + \underbrace{\beta_{n-1,n}\tilde{x}_n[i] + v_{n-1}[i]}_{\text{Is this bounded??}}$$

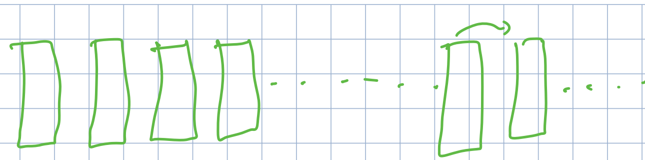
We know $|\lambda_{n-1}| < 1$

$$\begin{aligned} |\beta_{n-1,n}\tilde{x}_n[i] + v_{n-1}[i]| &\leq |\beta_{n-1,n}| \cdot |\tilde{x}_n[i]| + |v_{n-1}[i]| \\ &\leq |\beta_{n-1,n}| \cdot K_n \cdot \epsilon + \epsilon \\ &= (|\beta_{n-1,n}| \cdot K_n + 1)\,\epsilon \end{aligned}$$

$\Longrightarrow \exists K_{n-1}$ $|\tilde{x}_{n-1}[i]| \leq K_{n-1}\,\epsilon$

In particular $K_{n-1} = \left(\dfrac{1}{1 - |\lambda_{n-1}|}\right)\left((|\beta_{n-1,n}| \cdot K_n + 1)\right)$

Normal "Simple" Induction



1) First domino falls

2) If k^{th} domino falls, next one $(k+1)$ also falls

$(1) + (2) \implies$ all dominos fall.

Strong Induction

$\checkmark$ (1) First domino falls

$\checkmark$ (2') If all of the first k dominos fall, $(k+1)^{st}$ domino also falls

$(1) + (2') \implies$ all dominos fall.

(2') equivalent statement:

Assume $\forall k$ in $\ell+1, \ell+2, \ldots, n$

We know $\exists K_k$ s.t. $|\tilde{x}_k[i]| \le K_k \cdot \epsilon$

Want to show $\exists K_\ell$ s.t. $|\tilde{x}_\ell[i]| \le K_\ell \cdot \epsilon$

$$\tilde{x}_\ell[i+1] = \lambda_\ell \tilde{x}_\ell[i] + \underbrace{\left(\sum_{k=\ell+1}^{n} \beta_{\ell,k} \tilde{x}_k[i]\right) + v_\ell[i]}_{\text{Is this bounded?}}$$

Know $|\lambda_\ell| < 1$

$$\left|\left(\sum_{k=\ell+1}^{n} \beta_{\ell,k} \tilde{x}_k[i]\right) + v_\ell[i]\right| \le \sum_{k=\ell+1}^{n} |\beta_{\ell,k}| |\tilde{x}_k[i]| + |v_\ell[i]|$$

$$\le \sum_{k=\ell+1}^{n} |\beta_{\ell,k}| \cdot K_k \epsilon + \epsilon$$

$$\le \left(\left(\sum_{k=\ell+1}^{n} |\beta_{\ell,k}| \cdot K_k\right) + 1\right) \cdot \epsilon$$

$\implies \exists K_\ell$ s.t. $|\tilde{x}_\ell[i]| \le K_\ell \cdot \epsilon$

In particular $K_\ell = \left(\frac{1}{1 - |\lambda_\ell|}\right)\left(\left(\sum_{k=\ell+1}^{n} |\beta_{\ell,k}| \cdot K_k\right) + 1\right)$

This proves that $\exists\, K_k \quad \forall\, k$ in $1, 2, \dots, n$

$$\text{s.t.} \quad |\tilde{x}_k[i]| \le K_k \cdot \epsilon$$

$$\implies \exists\, K \quad \text{s.t.} \quad \|\vec{\tilde{x}}[i]\| \le K \cdot \epsilon$$

So:

$$\begin{array}{l} \text{bounded} \\ \\ i^{th}\text{ row} \\ \text{bounded} \\ \text{bounded} \\ \text{bounded} \end{array}
\begin{bmatrix} \tilde{x}_1[i+1] \\ \tilde{x}_2[i+1] \\ \vdots \\ \tilde{x}_{n-2}[i+1] \\ \tilde{x}_{n-1}[i+1] \\ \tilde{x}_n[i+1] \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & & \beta_{i,j} \\ & \lambda_2 & & & \\ & & \ddots & & \\ & 0 & & \lambda_{n-2} & \\ & & & \lambda_{n-1} & \beta_{n-1,n} \\ & & & & \lambda_n \end{bmatrix} \begin{bmatrix} \tilde{x}_1[i] \\ \tilde{x}_2[i] \\ \vdots \\ \vdots \\ \tilde{x}_{n-1}[i] \\ \tilde{x}_n[i] \end{bmatrix} + \begin{bmatrix} v_1[i] \\ \vdots \\ \vdots \\ \vdots \\ v_{n-1}[i] \\ v_n[i] \end{bmatrix}$$

This proves that having all eigenvalues $|\lambda_i| < 1 \implies$ BIBO stability

Technically, we only know that Upper-triangularization works when all the eigenvalues are real.

But there's a natural generalization to complex case.

<u>One more question</u>: What do we know about the $\beta_{i,j}$?

Could they be really huge??

$$\tilde{A} = U^T A U$$

Can entries of $\tilde{A}$ become huge?

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$$

First step: Understand $U^T A = \begin{bmatrix} U^T \vec{a}_1 & U^T \vec{a}_2 & \cdots & U^T \vec{a}_n \end{bmatrix}$

Consider $\| U^T \vec{a}_i \|^2 = \vec{a}_i^T \, U U^T \vec{a}_i = \vec{a}_i^T I \vec{a}_i$

$$= \vec{a}_i^T \vec{a}_i$$
$$= \| \vec{a}_i \|^2$$
$$= \sum_{j=1}^{n} a_{j,i}^2$$

$$\Longrightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} \left((U^T A)_{j,i} \right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{j,i}^2$$

__Definition:__ The Frobenius Norm of a matrix A is $\|A\|_F$ and defined by $\|A\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{j,i}^2$

We just proved: if $U^T U = I$ $\quad \|UA\|_F = \|A\|_F$

From the definition: $\|A^T\|_F = \|A\|$

$$\|\tilde{A}\|_F = \|U^T A U\|_F = \|AU\| = \|U^T A^T\|_F$$

since $UU^T = I$

$$= \|A^T\|_F$$
$$= \|A\|_F$$

$\Longrightarrow$ $\tilde{A}$ is no bigger than A.

$\Longrightarrow$ The β_{ij} aren't huge unless A was huge.

The style of our argument is useful beyond just BIBO stability

So the HW has you using it to solve differential equations.

Key: Upper-triangularization turns a vector/matrix problem into a cascade of scalar problems that allows counterparts of back-substitution to solve.