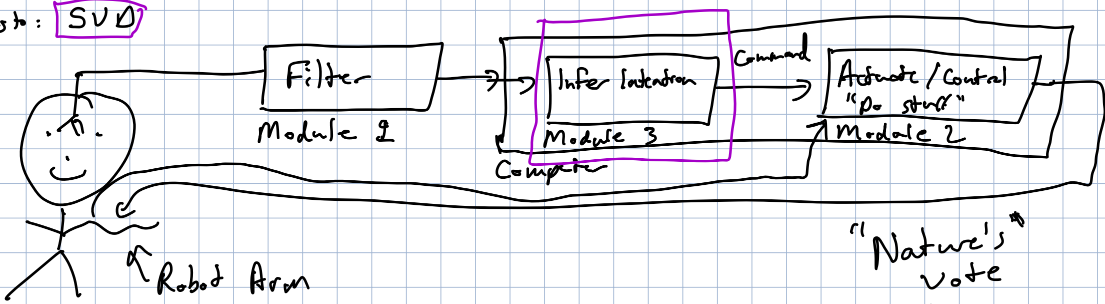
16B Prof. Anant Sahai

Today: Finish symmetric matrix case
Minimum Energy Control
↳ Solving $C\vec{u} = \vec{d}$ where C is wide.

Building to: $\boxed{SVD}$



Discrete model: $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i]$

Story so far: (1) Can use data to learn the system model using least squares
(2) To achieve our goals using this model, we need to:
(A) Make a plan for controls $\vec{u}[i]$ that achieve our goal
(B) Understand how to reliably execute that plan interactively in the face of disturbances.

Guess/Hope: If S is a symmetric real matrix, then all of its eigenvalues λ are purely real.

Symmetric: $S = S^T$

What do we know?

$S = S^T$

If $\lambda = \bar{\lambda}$, then λ is purely real.

$z + \bar{z}$ is real

$z\bar{z} = |z|^2$ is real.

$S\vec{v} = \lambda\vec{v}$ if λ is an eigenvalue and $\vec{v}$ is the corresponding eigenvector.

WLOG: Choose $\vec{v}$ to have unit norm $\|\vec{v}\| = 1$

Aside: You can brute-force the proof for 2×2 symmetric matrices.

$S = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ ← Exercise

$S\vec{v} = \lambda\vec{v} \Rightarrow \vec{v}^T S^T = \lambda\vec{v}^T$
$\Downarrow$
$\vec{v}^T S = \lambda\vec{v}^T$

$\overline{(S\vec{v})} = \bar{S}\bar{\vec{v}} = \overline{(\lambda\vec{v})}$

$\bar{S} = S$: $S\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$

We just learned that for real matrices, eigenvalues & eigenvectors come in conjugate pairs.

Want to isolate λ somehow. I know $\vec{v}^T S = \lambda \vec{v}^T$

and so $\vec{v}^T S \vec{v} = \lambda \vec{v}^T \vec{v}$
$\cancel{=} \lambda \cdot \|\vec{v}\|^2$?
$= \lambda$

Why is $\vec{v}^T \vec{v}$ not necessarily $\|\vec{v}\|^2$?

$$\sum_{i=1}^n v_i \cdot v_i \ne \sum_{i=1}^n |v_i|^2 \quad \text{if } v_i \text{ were complex.}$$

Second try: Consider $\vec{v}^T S \overline{\vec{v}} = \lambda \vec{v}^T \overline{\vec{v}}$

$$\vec{v}^T \overline{\vec{v}} = \sum_{i=1}^n v_i \overline{v}_i = \sum_{i=1}^n |v_i|^2$$

$= \lambda \|\vec{v}\|^2$
$= \lambda$

Suggests looking at $\vec{v}^T (S \overline{\vec{v}})$ alternatively.

$$\vec{v}^T \overline{\lambda} \overline{\vec{v}} = \overline{\lambda} \cdot \vec{v}^T \overline{\vec{v}}$$
$$= \overline{\lambda} \cdot \|\vec{v}\|^2 = \overline{\lambda}$$

I have shown $\lambda = \vec{v}^T S \overline{\vec{v}} = \overline{\lambda}$

So λ is purely real.

Spectral Theorem $\leftarrow$ Not a scary theorem.
From "Spectrum" = Rainbow.

If S is a symmetric real matrix, then it has a full complement of eigenvectors that are orthogonal and purely real that correspond to real eigenvalues $\lambda_1, \ldots, \lambda_n$.

$$S = V \Lambda V^T \quad \text{where} \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

where V is an orthonormal matrix

Given any real vector $\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$

Where $\alpha_i = \vec{v}_i^T \vec{x}$

$$(S)\vec{x} = \sum_{i=1}^{n} \alpha_i S \vec{v}_i = \sum_{i=1}^{n} \alpha_i \lambda_i \vec{v}_i$$

$$= \sum_{i=1}^{n} \lambda_i \vec{v}_i \alpha_i$$

$$= \left(\sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T\right) \vec{x}$$

For symmetric real matrices $S = \sum_{i=1}^{n} \lambda_i \underbrace{\vec{v}_i \vec{v}_i^T}_{\text{outer product}}$

Gives an $n \times n$ matrix.

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}\,u[i]$$

Start at $\vec{x}[0] = \vec{s}$

Want to get to a destination in T time steps.

$$\vec{x}[T] = \vec{g}$$

What controls $u[0], u[1], \ldots, u[T-1]$ should we use?

$$\vec{x}[T] = A^T \vec{x}[0] + A^{T-1}\vec{b}\,u[0] + A^{T-2}\vec{b}\,u[1] + \cdots + A b\, u[T-2] + \vec{b}\,u[T-1]$$

$$\underbrace{\begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{T-1}\vec{b} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} u[T-1] \\ u[T-2] \\ \vdots \\ u[0] \end{bmatrix}}_{\vec{u}} = \underbrace{\vec{g} - A^T \vec{s}}_{\vec{d}}$$

Want to solve $C\vec{u} = \vec{d}$ for $\vec{u}$

$$n\left\{\overbrace{\begin{bmatrix} \quad\quad C \quad\quad \end{bmatrix}}^{T}\right. \begin{bmatrix} \vec{u} \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{d} \end{bmatrix}\Big\} n$$

e.g. $n = 10$ ← 10 dim state.

$T = 100$ ← e.g. Want to reach $\vec{z}$ in One second and can act at 100 Hz.

← T-long,

Typical Problem: Infinitely many solutions. Which one do we want?

Principle of "being lazy": Pick the lowest energy solution.

$\Longrightarrow$ Minimum energy control.

Want $\underset{\vec{u}\in\mathbb{R}^T}{\arg\min} \|\vec{u}\|^2$ s.t. $C\vec{u} = \vec{d}$

Related in spirit to least-squares.

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} \vec{u} \end{bmatrix} \approx \begin{bmatrix} \\ \vec{d} \\ \\ \end{bmatrix}$$

$\underset{\vec{u}\in\mathbb{R}^T}{\arg\min} \|\vec{d} - A\vec{u}\|^2$

<u>How to solve???</u>

a) Invoke Calculus, <u>Need Math 53</u>
Have a set of n constraints
$\Longrightarrow$ Lagrange Multipliers, etc...

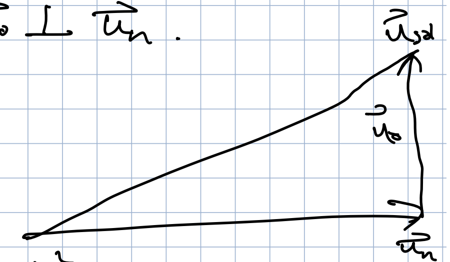
b) Think about the geometry & <u>pick the right basis.</u>

<u>Start Thinking</u> Suppose $\vec{u}_{sol}$ solved $C\vec{u}_{sol} = \vec{d}$

Know C is wide $\Longrightarrow$ It has a big nullspace.

Write: $\vec{u}_{sol} = \vec{u}_n + \vec{u}_o$ $\qquad \vec{u}_o \perp \vec{u}_n$.

$\vec{u}_n \in \text{Nullspace}(C)$



By Pythagoras: $\|\vec{u}_{sol}\|^2 = \|\vec{u}_n\|^2 + \|\vec{u}_o\|^2$

$$C\vec{u}_{sol} = \vec{d}$$

$$C(\vec{u}_n + \vec{u}_o) = \vec{0} + C\vec{u}_o \quad \Longrightarrow \quad C\vec{u}_o = \vec{d}$$

$\vec{u}_o$ would be a better solution than $\vec{u}_{sol}$.

$\Longrightarrow$ Want is a $\vec{u}$ that has no component in the nullspace of C.

We wish we had a basis where the nullspace of C was clearly visible.

Might as well wish for orthonormality of the basis too.

Imagine if we had such an orthonormal basis V.

First n columns of V are <u>NOT</u> in the nullspace of C

Last $T-n$ " " " are in the nullspace of C.

$$T\Big\{\underbrace{V}_{T} = \Big[\underbrace{V_1}_{n} \quad \underbrace{V_2}_{T-n}\Big]\Big\}\,T$$

V_2 is an orthonormal basis for the nullspace of C.

$V_1 \perp V_2$

Work in these nicer coordinates.

$$\vec{\tilde{u}} = V^T \vec{u} \implies \vec{u} = V \vec{\tilde{u}}$$

Coordinates in nice basis

original basis.

$$C\vec{u} = C V \vec{\tilde{u}}$$

$$= \begin{bmatrix} & C & \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \vec{\tilde{u}} \end{bmatrix}$$

$$= n\Big\{ \overbrace{\Big[\underbrace{CV_1}_{n} \;\Big|\; \underbrace{CV_2}_{J-n} \Big]}^{J} \begin{bmatrix} \vec{\tilde{u}} \end{bmatrix}$$

Recall $\|\vec{\tilde{u}}\| = \|\vec{u}\|$ by orthonormality of V.

$$= \Big[\overbrace{CV_1}^{n} \;\Big|\; \overbrace{0}^{J-n} \Big] \begin{bmatrix} \tilde{u}_{top} \\ \tilde{u}_{bot} \end{bmatrix} \begin{array}{l} \}\, n \\ \}\, J-n \end{array} = \begin{bmatrix} \vec{d} \end{bmatrix}$$

$$\|\tilde{u}\|^2 = \|\tilde{u}_{top}\|^2 + \|\tilde{u}_{bot}\|^2$$

$\implies$ Set $\tilde{u}_{bot} = \vec{0}$ and solve for $\tilde{u}_{top}$.

$$n\Big\{ \Big[\underbrace{CV_1}_{n} \Big] \vec{\tilde{u}}_{top} = \vec{d}$$

Invert.

How can I find an orthonormal basis s.t. Nullspace of C is spanned by the last basis elements

Can we use the spectral theorem?

I would need a symmetric matrix S with the same nullspace as C.

$$T \; \boxed{S} \; T = \, ? \; C$$

If $S = ? \cdot C$ then $\vec{u} \in \text{Nullspace}(C)$
$S\vec{u} = \vec{0}$

$$? \, C = (? \, C)^T$$
$$= C^T \, ?$$

Try $S = C^T C$

Does $C^T C$ have the same nullspace as C?

$$\vec{u} \in \text{Nullspace}(C) \implies \vec{u} \in \text{Nullspace}(S)$$

Need $\vec{u} \in \text{Nullspace}(S) \implies \vec{u} \in \text{Nullspace}(C)$

$$S\vec{u} = \vec{0} \implies C^T C \vec{u} = \vec{0}$$
$$\implies \vec{u}^T C^T C \vec{u} = 0$$

$$= (C\vec{u})^T (C\vec{u}) = 0$$

$$= \| C\vec{u} \|^2 = 0$$

$$= C\vec{u} = \vec{0}$$

So $S = C^T C$ <u>works!</u>