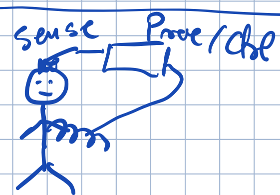
Lecture 4

* Sensing
  * Non-homogeneous diff. eqns.
    & with continuous inputs



signals of interest

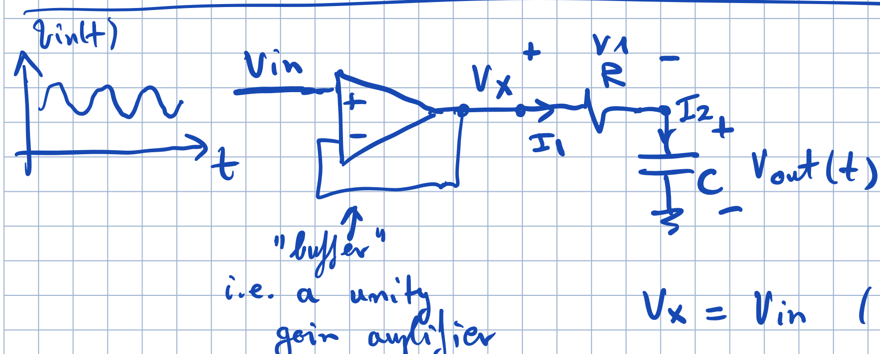
retain

+ ~~interference (AC power, wifi, cellular, FM, AM ....)~~

filter-out

Goal: Filter (select) signals of interest

How can a circuit do this "computation"?



$V_x = V_{in}$ (buffer)

KCL: $I_1 = I_2$

$V_1 = V_x - V_{out}$

$\left.\begin{aligned} V_1 &= I_1 \cdot R \\ I_2 &= C \cdot \frac{dV_{out}}{dt} \end{aligned}\right\}$ Elements

$$\frac{V_{in} - V_{out}}{R} = C\frac{dV_{out}}{dt}$$

NVA
nodal voltage analysis

$$\frac{dV_{out}(t)}{dt} = -\frac{V_{out}(t)}{RC} + \boxed{\frac{V_{in}(t)}{RC}}$$

continuous input ?

In disc. + hw:

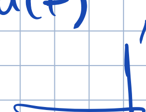
$$V_{out}(t) = V_{out}(0)\, e^{-\frac{t}{RC}} + \underbrace{\frac{1}{RC}\int_0^t v_{in}(\theta)\, e^{-\frac{1}{RC}(t-\theta)}\, d\theta}$$

homogeneous solution
"response to init. condition"

non-homogeneous soln.
"response to time-varying input"

The circuit "computes" this solution ?

General form: $\frac{d}{dt}x(t) = \lambda x(t) - \lambda u(t)$, $\lambda = -\frac{1}{RC}$



$u(t) = v_{in}(t)$

Let's try to "compute" the response for some $u(t)$'s of interest.

Exomple 1: $u(t) = e^{st}$

$$(1) \quad \boxed{\frac{d}{dt} x(t) = \lambda x(t) - \lambda u(t)}, \quad t \geq 0$$

$$\boxed{s \neq \lambda}$$

can solve:

$$x(t) = x(0) e^{\lambda t} - \lambda \int_0^t u(\theta) e^{\lambda(t-\theta)} d\theta$$

$$= x(0) e^{\lambda t} - \lambda e^{\lambda t} \int_0^t \underbrace{e^{s\theta} \cdot e^{-\lambda\theta}}_{e^{(s-\lambda)\theta}} d\theta$$

$\vdots$

or Guess & check:

$$x(t) = K \cdot e^{st} \quad , t \geq 0$$

From (1)
$$K \cdot s \cdot e^{st} = \lambda \cdot K e^{st} - \lambda \cdot e^{st}$$
$$Ks = \lambda K - \lambda$$
$$K = -\frac{\lambda}{s-\lambda} \Rightarrow x(t) = \frac{-\lambda}{s-\lambda} \cdot e^{st}$$

To complete, add a homogeneous soln.

$$K_2 e^{\lambda t}$$

$$x(t) = K_2 e^{\lambda t} + \underbrace{K \cdot e^{st}}$$

$$x(0) = K_2 + K \Rightarrow K_2 = x(0) - K$$

$$\boxed{x(t) = \left(x(0) + \frac{\lambda}{s-\lambda}\right)e^{\lambda t} - \frac{\lambda}{s-\lambda} e^{st}}$$

Example 2: $u(t) = \cos(\omega t)$

$$\frac{d}{dt} x(t) = \lambda x(t) - \lambda u(t)$$

Our RC system $\lambda = -\frac{1}{RC}$

$$\boxed{x(t) = x(0) e^{\lambda t} - \lambda \int_0^t u(\theta) e^{\lambda(t-\theta)} d\theta}$$

Formula

$$= x(0) e^{\lambda t} - \lambda e^{\lambda t} \int_0^t u(\theta) e^{-\lambda \theta} d\theta$$

$$= x(0) e^{\lambda t} - \lambda e^{\lambda t} \int_0^t \cos(\omega \theta) e^{-\lambda \theta} d\theta$$

$$\boxed{\int \cos(bx) \cdot e^{ax} dx = \frac{e^{ax}}{a^2+b^2}\left(b \sin(bx) + a \cos(bx)\right)}$$

$$x(t) = x(0)e^{\lambda t} - \lambda e^{\lambda t}\left(\frac{e^{-\lambda t}}{\lambda^2+\omega^2}\left(\omega \sin(\omega t) - \lambda \cos(\omega t)\right) - \frac{1}{\lambda^2+\omega^2}\left(0-\lambda\right)\right)$$

$$= x(0)e^{\lambda t} - \frac{\lambda}{\lambda^2+\omega^2}\left(\omega \sin(\omega t) - \lambda \cos(\omega t)\right) - \frac{\lambda^2 e^{\lambda t}}{\lambda^2+\omega^2}$$

$$\boxed{x(t) = \underbrace{\left(x(0) - \frac{\lambda^2}{\lambda^2+\omega^2}\right)e^{\lambda t}}_{①} - \underbrace{\frac{\lambda\left(\omega \sin(\omega t) - \lambda \cos(\omega t)\right)}{\lambda^2+\omega^2}}_{②}}$$

$t \to \infty$ (steady-state)

② $\lambda < 0$ $\quad \lambda = -\frac{1}{RC}$

① $\to 0$ b.c. $\lambda < 0$, $\lambda = -\frac{1}{RC}$

② $\lambda = -\frac{1}{RC}$

$$x(t) = \frac{\frac{1}{RC} \cdot \omega \sin(\omega t) + \left(\frac{1}{RC}\right)^2 \cos(\omega t)}{\left(\frac{1}{RC}\right)^2 + \omega^2}$$

$$= \boxed{\frac{\omega RC \sin(\omega t) + \cos(\omega t)}{1 + (\omega RC)^2}}$$

Remember:

$u(t) = \cos(\omega t)$

ω - angular frequency

$\omega = \frac{2\pi}{T} = 2\pi \cdot f$

T - period

Case 1: $\omega >> \frac{1}{RC}$

$$\Downarrow$$

$$\omega RC >> 1$$

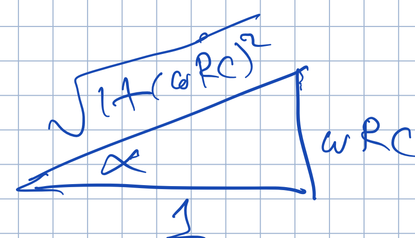
$$x(t) \simeq 0$$

Case 2: $\omega << \frac{1}{RC}$

$$\Downarrow$$

$$\omega RC << 1$$

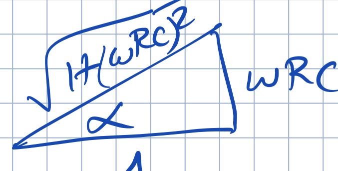
$$x(t) = \cos(\omega t + \theta)$$



$\alpha = \text{atan2}(\omega RC, 1)$

"low-pass" filter because it "passes" the low frequencies and attenuates the high frequencies (w.r.t. $\frac{1}{RC}$)

$\alpha = \text{atan2}(\omega RC, 1)$



$$x(t) = \frac{1}{\sqrt{1+(\omega RC)^2}} \left(\underbrace{\frac{1}{\sqrt{1+(\omega RC)^2}}}_{\cos(\alpha)} \cos(\omega t) + \underbrace{\frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}_{\sin(\alpha)} \sin(\omega t) \right)$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}} \left(\cos(\alpha) \cos(\omega t) + \sin(\alpha) \sin(\omega t) \right)$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}} \cos(\omega t - \alpha)$$

$\theta = -\alpha$

$\theta = -\mathrm{atan2}(\omega RC, 1)$

$$\boxed{x(t) = \frac{1}{\sqrt{1+(\omega RC)^2}} \cdot \cos(\omega t + \theta)}$$

Can also arrive here with a guess:

Guess: $x(t) = A \cos(\omega t + \theta)$ $\qquad u(t) = V_{in} \cos(\omega t)$

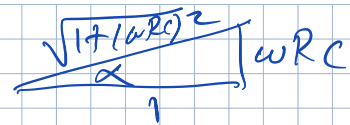
System: $\frac{d}{dt} x(t) = \lambda x(t) - \lambda u(t)$

$$-A\omega \sin(\omega t + \theta) = \lambda \cdot A \cos(\omega t + \theta) - \lambda V_{in} \cos(\omega t)$$

$$V_{in} \cos(\omega t) = A \cos(\omega t + \theta) + \frac{A\omega}{\lambda} \sin(\omega t + \theta)$$

$$= A \left(\cos(\omega t + \theta) + \frac{\omega}{\lambda} \sin(\omega t + \theta) \right)$$

$$\lambda = -\frac{1}{RC}$$



$$\alpha = \operatorname{atan2}(\omega RC, 1)$$

$$V_{in} \cos(\omega t) = A\left(1 \cdot \cos(\omega t + \theta) - \omega RC \sin(\omega t + \theta)\right)$$

$$V_{in} \cos(\omega t) = A\sqrt{1+(\omega RC)^2}\left(\overbrace{\frac{1}{\sqrt{1+(\omega RC)^2}}}^{\cos(\alpha)} \cdot \cos(\omega t + \theta) - \overbrace{\frac{\omega RC}{\sqrt{1+(\omega RC)^2}}}^{\sin(\alpha)} \cdot \sin(\omega t + \theta)\right)$$

$$V_{in} \cos(\omega t) = A\sqrt{1+(\omega RC)^2} \cos(\omega t + \theta + \alpha)$$

$$V_{in} = A\sqrt{1+(\omega RC)^2} \Rightarrow \boxed{A = \frac{V_{in}}{\sqrt{1+(\omega RC)^2}}}$$

$$\theta + \alpha = 0 \Rightarrow \boxed{\theta = -\alpha = -\operatorname{atan2}(\omega RC, 1)}$$

$$\left[\cos\alpha \cos\beta - \sin\alpha \sin\beta = \cos(\alpha + \beta)\right]$$

$$x(t) = \boxed{V_{out}(t) = \frac{V_{in}}{\sqrt{1+(\omega RC)^2}} \cdot \cos(\omega t + \theta)}$$

more general:

$$x(t) = x(t_0)\, e^{\lambda \cdot (t - t_0)} - \lambda \int_{t_0}^{t} u(\theta)\, e^{\lambda (t - \theta)}\, d\theta$$

$$t \geq t_0$$