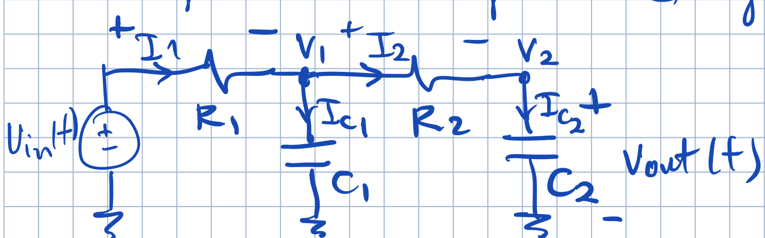
Lecture 5

* Systems of differential equations
  * Higher-order d.e.s
  * Circuits w. multiple c's
  * Vector DEs
* Diagonalization

Going to more complex systems - circuits example
simplest two-capacitor example:



KCL: $I_1 = I_2 + I_{C_1}$

$I_2 = I_{C_2}$

$V_{in} - V_1 = I_1 \cdot R_1$

$V_{out} = V_2 - 0 = V_2$

$V_1 - V_2 = I_2 \cdot R_2$

$I_{C_1} = C_1 \dfrac{dV_1}{dt}$, $I_{C_2} = C_2 \dfrac{dV_2}{dt}$

$I_1 = I_{C_2} + I_{C_1}$

from NVA directly

(1) $\dfrac{V_{in} - V_1}{R_1} = C_1 \dfrac{dV_1}{dt} + C_2 \dfrac{dV_2}{dt}$

(2) $\dfrac{V_1 - V_2}{R_2} = C_2 \dfrac{dV_2}{dt} \Rightarrow \boxed{V_1 = V_2 + R_2 C_2 \dfrac{dV_2}{dt}}$

$$R_1C_1R_2C_2\frac{d^2V_2}{dt} + (R_1C_1 + R_1C_2 + R_2C_2)\frac{dV_2}{dt} + V_2 - V_{in} = 0$$

(2^{nd} order diff. equation)

don't know how to solve it yet

$$① \quad \frac{V_{in}-V_1}{R_1} = C_1\frac{dV_1}{dt} + \frac{V_1-V_2}{R_2}$$

$$② \quad \frac{V_1-V_2}{R_2} = C_2\frac{dV_2}{dt}$$

$$① \quad \frac{dV_1}{dt} = -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_1 + \frac{V_2}{R_2C_1} + \frac{V_{in}}{R_1C_1}$$

$$② \quad \frac{dV_2}{dt} = \frac{1}{R_2C_2}V_1 - \frac{1}{R_2C_2}V_2$$

$$\frac{d}{dt}\begin{bmatrix}V_1\\V_2\end{bmatrix} = \begin{bmatrix}-\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1}\\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2}\end{bmatrix}\begin{bmatrix}V_1\\V_2\end{bmatrix} + \begin{bmatrix}\frac{1}{R_1C_1}\\0\end{bmatrix}V_{in}$$

only know how to solve

$$\frac{d}{dt}x(t) = \lambda x + u$$

Example: Assume: $R_1 = \frac{1}{3} M\Omega$ ($M = 10^6$), $R_2 = \frac{1}{2} M\Omega$;

$C_1 = C_2 = 1 \mu F$ ($\mu = 10^{-6}$)

(a) $$\boxed{\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}}$$

How about some "magic":

(b) $u_1 = V_2$

(c) $u_2 = V_1 + 2V_2$

[Assume: $V_{in} = 1V$ $t < 0$

$0V$ $t \geq 0$

$V_1(0) = 1V$ & $V_2(0) = 1V$

solving for $t \geq 0$]

$$\frac{d}{dt}u_1 \overset{(b)}{=} \frac{d}{dt}V_2 \overset{(a)}{=} 2V_1 - 2V_2 \overset{(c)}{=} 2(u_2 - 2V_2) - 2V_2$$

$$= 2u_2 - 4V_2 - 2V_2 = 2u_2 - 6V_2 = 2u_2 - 6u_1$$

(d) $$\boxed{\frac{d}{dt}u_1 = 2u_2 - 6u_1}$$

$$\frac{d}{dt}u_2 \overset{(c)}{=} \frac{d}{dt}(V_1 + 2V_2) = \frac{d}{dt}V_1 + 2\frac{d}{dt}V_2 \overset{(a)}{=}$$

$$= -5V_1 + 2V_2 + 2(2V_1 - 2V_2) =$$

$$= -5V_1 + 2V_2 + 4V_1 - 4V_2 =$$

$$= -V_1 - 2V_2 \overset{(c)}{=} -u_2$$ Wohoo!

(e) $$\boxed{\frac{d}{dt}u_2 = -u_2}$$ (homogeneous) Know how to solve u_2.

$$\Rightarrow \quad u_2(t) = u_2(0)\, e^{-t} \quad , \quad t \geq 0$$

$$u_2(0) \overset{(c)}{=} V_1(0) + 2V_2(0) = 1V + 2 \cdot 1V = 3V$$

(f) $\boxed{u_2(t) = 3 \cdot e^{-t} \quad , \quad t \geq 0}$

(d) $\frac{d}{dt} u_1 = 2u_2 - 6u_1$

$$\frac{d}{dt} u_1 = -6u_1 + 6e^{\overset{s=-1}{st}} \quad , \quad t \geq 0$$

$\lambda = -6$

$s = -1$

[Remember:

$\frac{d}{dt} x(t) = \lambda x(t) - \lambda u(t)$

where $u(t) = e^{st}$

$x(t) = k_2 e^{\lambda t} - \frac{\lambda}{s-\lambda} e^{st}$

$x(0) = k_2 + \frac{\lambda}{s-\lambda}$]

$u_1(t) = k_2 \cdot e^{-6t} + \frac{6e^{-t}}{-1+6}$

$u_1(0) = k_2 + \frac{6}{5}$

$u_1(0) \overset{(b)}{=} V_2(0) = 1V \Rightarrow k_2 = -\frac{1}{5}$

(g) $\boxed{u_1(t) = -\frac{1}{5} e^{-6t} + \frac{6}{5} e^{-t}}$

have solved for $u_1(t)$ & $u_2(t)$ $\Rightarrow$

$\Rightarrow$ back-solve for $\underline{V_1(t) \text{ \& } V_2(t)}$

$$v_1(t) \overset{(b)\&(c)}{=} u_2(t) - 2u_1(t) \overset{(f)\&(g)}{=}$$

$$= 3\cdot e^{-t} - 2\left(-\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}\right)$$

$$= 3\cdot e^{-t} + \frac{2}{5}e^{-6t} - \frac{12}{5}e^{-t}$$

(h) $\boxed{v_1(t) = \frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t}}$

(i) $\boxed{v_2(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}}$

(a) $$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} v_{in}$$

(b) & (c) $$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \overset{(d)}{\Longleftrightarrow} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}}_{W}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{d}{dt}\left(W^{-1}\cdot\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = W^{-1}\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$$

$$\overset{(a)}{=} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}}\cdot\left(\underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} v_{in}\right)$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \overset{(f)}{=} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}}_{W} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} +$$

$$+ \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \cdot \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} v_{in}$$

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix}}_{W^{-1}AW} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{W^{-1}\vec{b}} v_{in}$$

upper-triangular

$\Rightarrow$ so, we can peel-back the solutions through GE.

Summary: Systems of differential equations

(1) $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$, $\vec{x}(0)$

$\vec{x}(t) = ?$

for $t \geq 0$.

Native $\vec{x}$ coordinates: Variables derivatives

"Nice" coordinates $\vec{\tilde{x}}$:

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, "V^{-1}"'] \\
\vec{\tilde{x}} \arrow[u, "V"'] \arrow[r, "V^{-1}AV"'] & \frac{d}{dt}\vec{\tilde{x}} \arrow[u, bend right, "V"']
\end{tikzcd}$$

(2) $\vec{x}(t) = V\vec{\tilde{x}}(t)$

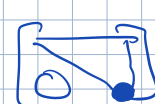
(3) $\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \frac{d}{dt}\left(V^{-1}\vec{x}(t)\right) = V^{-1}\frac{d}{dt}\vec{x}(t) \stackrel{(1)}{=}$$

$$= V^{-1}\left(A\vec{x}(t) + Bu(t)\right) = V^{-1}A\vec{x}(t) + V^{-1}Bu(t) \stackrel{(2)}{=}$$

$$= V^{-1}AV\vec{\tilde{x}}(t) + V^{-1}Bu(t)$$

(4) $$\boxed{\frac{d}{dt}\vec{\tilde{x}}(t) = \underbrace{V^{-1}AV}_{\text{want this matrix to be "nice"}}\vec{\tilde{x}}(t) + V^{-1}Bu(t)}$$

e.g. upper-triangular $\Rightarrow$ G.E. 

homogen.

or even better $\Rightarrow$ diagonal

 homogeneous. $\Rightarrow$ know how to solve.