Lecture 6

* Systems of differential equations
* Diagonalization

Lab-sim sections – expanding hybrid sections
join us & be done with lab in <3hrs

Have system: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$, $t \geq 0$

$\vec{x}(0)$

Original coordinates:

Nice coordinates:

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\
\tilde{\vec{x}} \arrow[r, "?"] \arrow[u, bend right, "V"'] & \frac{d}{dt}\tilde{\vec{x}} \arrow[u, bend right, "V"']
\end{tikzcd}$$

$\tilde{\vec{x}} = V^{-1}\vec{x}$ $\qquad$ $\vec{x} = V\tilde{\vec{x}}$

$$\frac{d}{dt}\tilde{\vec{x}} = \frac{d}{dt}(V^{-1}\vec{x}) = V^{-1}\frac{d}{dt}\vec{x} = V^{-1}(A\vec{x} + B\vec{u})$$

$$= V^{-1}AV\tilde{\vec{x}} + V^{-1}B\vec{u}$$

Step 1

$$\boxed{\frac{d}{dt}\tilde{\vec{x}}(t) = V^{-1}AV\tilde{\vec{x}}(t) + V^{-1}B\vec{u}(t)}$$

Step 2 Get $\tilde{\vec{x}}(0)$ from $\tilde{\vec{x}}(0) = V^{-1}\vec{x}(0)$
& solve for $\tilde{\vec{x}}(t)$ when

$V^{-1}AV$ is diagonal or upper-triag.

Step 3 Go back to $\vec{x}(t)$

$$\vec{x}(t) = V\vec{\tilde{x}}(t)$$

Want: $V^{-1}AV = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ (Diagonal matrix) $\Downarrow$ will give us homogenous solutions of scalar diff. eqns)

How to figure out which V to use?

$V = [\vec{v_1} \; \dots \; \vec{v_n}]$

$\uparrow$ From 26A:

$A\vec{v_i} = \lambda_i \vec{v_i}$

$\uparrow$ eigenvectors $\uparrow$ eigenvalues

$$V^{-1}AV = V^{-1}(A[\vec{v_1} \dots \vec{v_n}])$$
$$= V^{-1}[A\vec{v_1} \; \dots \; A\vec{v_n}]$$
$$= V^{-1}[\lambda_1\vec{v_1} \; \dots \; \lambda_n\vec{v_n}]$$

If $\vec{v_i}$'s are linearly indep and each eigenvector of A.

$$[\lambda_1\vec{v_1} \; \dots \; \lambda_n\vec{v_n}] =$$

$$= [\vec{v_1} \; \dots \; \vec{v_n}] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = V \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$V^{-1}AV = V^{-1}\left[\lambda_1 \vec{v_1} \; \cdots \; \lambda_n \vec{v_n}\right]$$

$$= \underbrace{V^{-1}V}_{I} \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = I \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda \text{ or } A_\lambda}$$

$V^{-1}AV = \Lambda$ - diagonal ☺

So, if V is an eigenbasis (basis of eigenvectors) then:

$$V^{-1}AV = \Lambda \quad \text{- diagonal}$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \underbrace{V^{-1}AV}_{\Lambda}\, \vec{\tilde{x}}(t) + V^{-1}B\vec{u}(t)$$

$$\boxed{\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\, \vec{\tilde{x}}(t) + V^{-1}B\vec{u}(t)}$$

a collection of scalar D.E.s ☺

Want to find eigenvalues (λ_i's) & eigenvectors ($\vec{v_i}$'s) of A to compose V and Λ.

Example: use our 2^{nd} order RC circuit example.

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

[Recall: $A\vec{v} = \lambda\vec{v}$

$$\underbrace{(A - \lambda I)}_{\text{has a nullspace}}\vec{v} = 0$$

$\det(A - \lambda I) = 0$ or

$\det(\lambda I - A) = 0$

if λ is an eigenvalue]

(1) Find eigenvalues: $\det(\lambda I - A)$

$$\det\left(\begin{bmatrix} \lambda+5 & -2 \\ -2 & \lambda+2 \end{bmatrix}\right)$$

$$= (\lambda+5)(\lambda+2) - 4 = \lambda^2 + 7\lambda + 6 =$$

$$= (\lambda+6)(\lambda+1) = 0$$

$$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = -6$$

$$\boxed{\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}}$$

(2) Find eigenvectors:

$A - \lambda I$ Null-space for $A - \lambda_1 I$?

$\lambda_1 = -1$

$$A-\lambda_1 I = \begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}}_{A-\lambda_1 I} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}_1} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}} \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\lambda_2 = -6$ Null-space of $A-\lambda_2 I = ?$

$$(A-\lambda_2 I)\,\vec{v}_2 = \vec{0}$$

$$A-\lambda_2 I = \begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A-\lambda_2 I} \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\vec{v}_2} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}} \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

$$\boxed{V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}$$

$$\boxed{V^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}}$$

③ Homogenous solution for $\vec{\hat{x}}(t)$:

$$\frac{d}{dt}\vec{\hat{x}}(t) = \underbrace{\Lambda}_{V^{-1}AV}\vec{\hat{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}\underbrace{\vec{\hat{x}}(t)}_{\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}} = \begin{bmatrix} -\hat{x}_1(t) \\ -6\hat{x}_2(t) \end{bmatrix}$$

$$\boxed{\frac{d}{dt}\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\hat{x}_1(t) \\ -6\hat{x}_2(t) \end{bmatrix}}$$

$$\Downarrow$$

$$\left\{ \frac{d}{dt}\hat{x}_1(t) = -\hat{x}_1(t), \quad \frac{d}{dt}\hat{x}_2(t) = -6\hat{x}_2(t) \right.$$

scalar D.Es.

$$\hat{x}_1(t) = \hat{x}_1(0)\cdot e^{-t}$$
$$\hat{x}_2(t) = \hat{x}_2(0)\cdot e^{-6t}$$

$$\vec{\hat{x}}(t) = \begin{bmatrix} \hat{x}_1(0)\,e^{-t} \\ \hat{x}_2(0)\,e^{-6t} \end{bmatrix}$$

a solution in eigenbasis coordinates.

Important: $\boxed{\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0)}$

$$\vec{\tilde{x}}(0) = \underbrace{\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}}_{V^{-1}} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}(0)} = \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \end{bmatrix}$$

$$\boxed{\vec{\tilde{x}}(t) = \begin{bmatrix} \frac{3}{5} \cdot e^{-t} \\ -\frac{1}{5} e^{-6t} \end{bmatrix}}$$, but want $\vec{x}(t)$

(4) $\vec{x}(t) = V \vec{\tilde{x}}(t) =$

$$= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5} e^{-t} \\ -\frac{1}{5} e^{-6t} \end{bmatrix} =$$

$$\boxed{\vec{x}(t) = \begin{bmatrix} \frac{3}{5} e^{-t} + \frac{2}{5} e^{-6t} \\ \frac{6}{5} e^{-t} - \frac{1}{5} e^{-6t} \end{bmatrix}}$$

(homogenous solution)

$V_{in} = 0V$ for $t \geq 0$ $\Rightarrow$ no non-homogen. soln.

In general: $\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\,\vec{\tilde{x}}(t) + \underbrace{V^{-1}B}_{\tilde{B}}\,\vec{u}(t)$

$\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\,\vec{\tilde{x}}(t) + \vec{\tilde{u}}(t)$ $\qquad \vec{\tilde{x}}(t) = \vec{\tilde{x}}_h(t) + \vec{\tilde{x}}_{nh}(t)$

e.g. 2×2

$$\vec{\tilde{x}}_{nh}(t) = \begin{bmatrix} \int_0^t \tilde{u}_1(\theta)\, e^{\lambda_1(t-\theta)}\, d\theta \\ \int_0^t \hat{u}_2(\theta)\, e^{\lambda_2(t-\theta)}\, d\theta \end{bmatrix}$$

non-homogenous solution

$\tilde{B} = \begin{bmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{bmatrix} = V^{-1}B$

[Remember the scalar case: $\frac{d}{dt}z(t) = \lambda z(t) + u(t) \Rightarrow$
$\Rightarrow z(t) = z(0)e^{\lambda t} + \int_0^t u(\theta)e^{\lambda(t-\theta)}d\theta$]

$$\vec{\tilde{u}} = V^{-1}B\cdot\vec{u} = \begin{bmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \hat{b}_{11}u_1(t) + \hat{b}_{12}u_2(t) \\ \hat{b}_{21}u_1(t) + \hat{b}_{22}u_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix}$$

For our 2^{nd} order RC example:

$\underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{B=\vec{b}} \cdot v_{in}(t)$ $\qquad \vec{\hat{u}} = V^{-1}BV_{in} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}$

$$\vec{\tilde{u}} = \begin{bmatrix} \frac{3}{5}v_{in}(t) \\ -\frac{6}{5}v_{in}(t) \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$$

$$\vec{\tilde{X}}_{nh}(t) = \begin{bmatrix} \int_0^t \frac{3}{5} v_{in}(\theta) e^{-1(t-\theta)} d\theta \\ \int_0^t \left(-\frac{6}{5}\right) v_{in}(\theta) e^{-6(t-\theta)} d\theta \end{bmatrix}$$

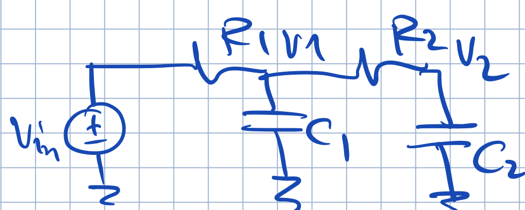
$$\vec{x}_{nh}(t) = V \vec{\tilde{x}}_{nh}(t)$$

$$V = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\vec{x}_{nh}(t) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} e^{-t} \int_0^t v_{in}(\theta) e^{\theta} d\theta \\ -\frac{6}{5} e^{-6t} \int_0^t v_{in}(\theta) e^{6\theta} d\theta \end{bmatrix}$$

$$\vec{x}_{nh}(t) = \begin{bmatrix} \frac{3}{5} e^{-t} \int_0^t v_{in}(\theta) e^{\theta} d\theta + \frac{12}{5} e^{-6t} \int_0^t v_{in}(\theta) e^{6\theta} d\theta \\ \frac{6}{5} e^{-t} \int_0^t v_{in}(\theta) e^{\theta} d\theta - \frac{6}{5} e^{-6t} \int_0^t v_{in}(\theta) e^{6\theta} d\theta \end{bmatrix}$$

New example:



For $v_{in}(t) = 0V \; t<0$
$= 1V \; t \geq 0$

$$\vec{x}(0) = \begin{bmatrix} v_1[0] \\ v_2[0] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_h(t) = 0$$

I.C. $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\vec{x}_{nh}(t) = \begin{bmatrix} \frac{3}{5}e^{-t}\int_0^t e^{\theta}d\theta + \frac{12}{5}e^{-6t}\int_0^t e^{6\theta}d\theta \\ \frac{6}{5}e^{-t}\int_0^t e^{\theta}d\theta - \frac{6}{5}e^{-6t}\int_0^t e^{6\theta}d\theta \end{bmatrix}$$

(non-homogenous solution)

$$= \begin{bmatrix} \frac{3}{5}e^{-t}(e^{t}-1) + \frac{12}{5}e^{-6t}\frac{1}{6}(e^{6t}-1) \\ \frac{6}{5}e^{-t}(e^{t}-1) - \frac{6}{5}e^{-6t}\frac{1}{6}(e^{6t}-1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} - \frac{3}{5}e^{-t} + \frac{2}{5} - \frac{2}{5}e^{-6t} \\ \frac{6}{5} - \frac{6}{5}e^{-t} - \frac{1}{5} + \frac{1}{5}e^{-6t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{3}{5}e^{-t} - \frac{2}{5}e^{-6t} \\ 1 - \frac{6}{5}e^{-t} + \frac{1}{5}e^{-6t} \end{bmatrix}$$

For $\vec{x}_h(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \vec{x}_h(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_{nh}(t)$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 - \frac{3}{5}e^{-t} - \frac{2}{5}e^{-6t} \\ 1 - \frac{6}{5}e^{-t} + \frac{1}{5}e^{-6t} \end{bmatrix}$$