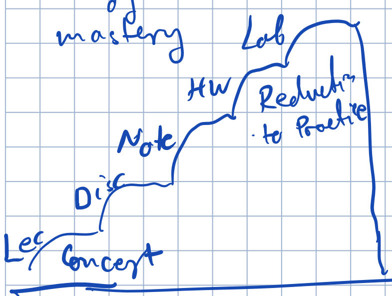
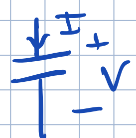
Lecture 7

16B Mountain of mastery



* Intro to inductors
* 2nd order systems with complex eigenvalues

Capacitor



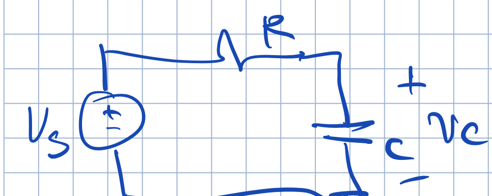
$$\boxed{I(t) = C\frac{d}{dt}V(t)}$$

* "resists a change in voltage"
* stores energy in electric field

$$\boxed{E = \frac{1}{2}CV^2}$$

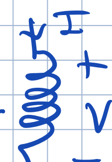
C − unit Farad [F]

* at DC: acts as an open circuit



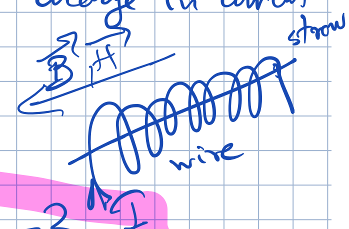
$v_c \sim e^{-\frac{t}{RC}}$, $\tau = RC$ time-constant

Inductor



$$\boxed{V(t) = L\frac{d}{dt}I(t)}$$

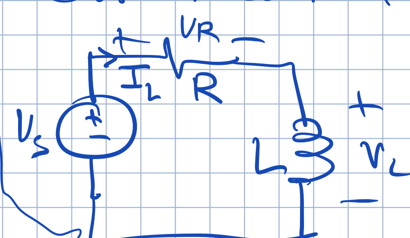
* "resists a change in current"
* Stores energy in magnetic field



$$\boxed{E = \frac{1}{2}LI^2}$$

* short-circuit at constant current

L − unit Henry [H]



$\tau = ?$

Decaying exponentials $e^{-\frac{t}{\tau}}$?

$V_s(t<0) = 1V$ Solve for $I_L(t)$.

$V_s(t\geq 0) = 0V$

First, find $I_L(0)$.

for $\underline{t<0}$

$$I_L(t) = \frac{\overset{1V}{V_s(t)} - \cancel{V_L(t)}}{R} = \frac{1V}{R}, \quad V_L(t) \to 0 = L\frac{dI}{dt} \ (I \text{ constant})$$

$$I_L(0) = \frac{1V}{R}$$

$$V_L(t) = L\frac{dI_L(t)}{dt}$$

$$I_L(t) = \frac{V_s(t) - V_L(t)}{R} = \frac{V_s(t)}{R} - \frac{L}{R}\frac{d}{dt}I_L(t)$$

For $V_s(t\geq 0) = 0V$

$$I_L(t) = -\frac{L}{R}\frac{d}{dt}I_L(t)$$

$$\boxed{\frac{d}{dt}I_L(t) = -\frac{R}{L}I_L(t)}, \quad t\geq 0$$

compare this to

$$\boxed{\frac{d}{dt}V_c(t) = -\frac{1}{RC}V_c(t)}$$

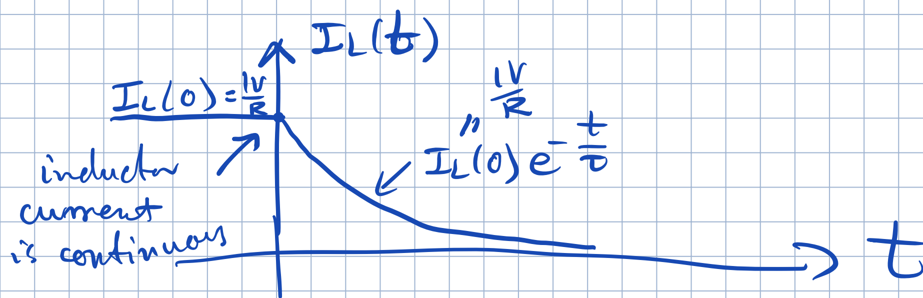
$$I_L(t) = I_L(0)\cdot e^{-\frac{R}{L}t}$$

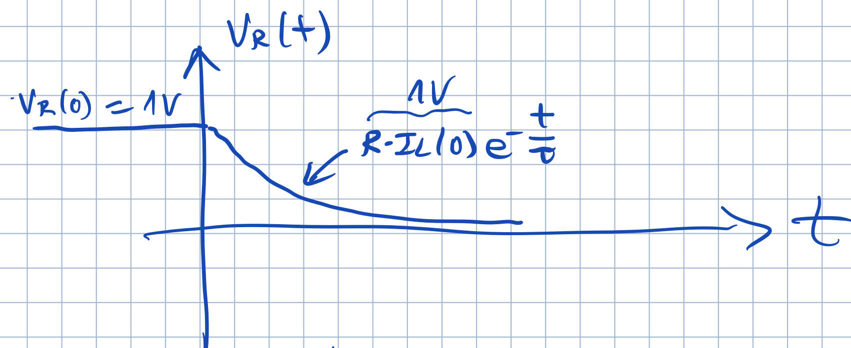
$$= I_L(0)\, e^{-\frac{t}{\tau}}, \quad \tau = \frac{L}{R} \text{ - time constant}$$

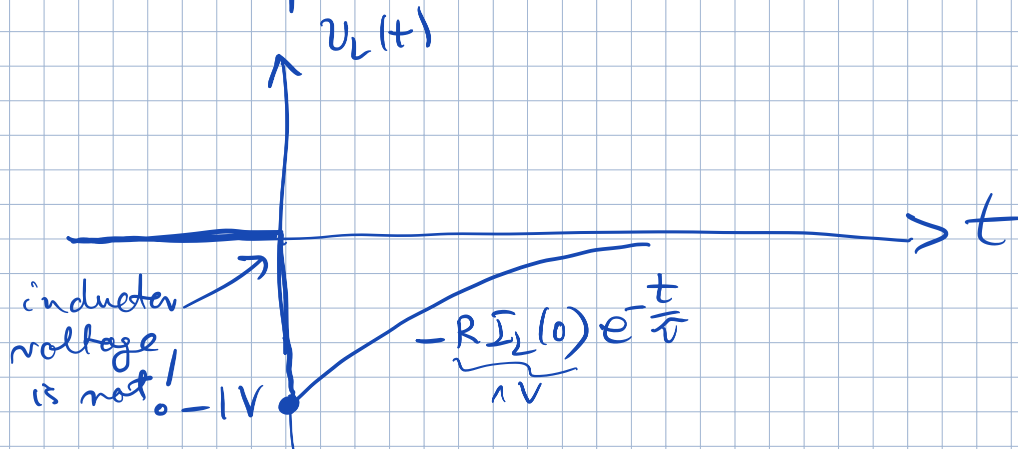
$$V_R(t) = R\cdot I_L(t) = R\cdot I_L(0)\, e^{-\frac{t}{\tau}}$$

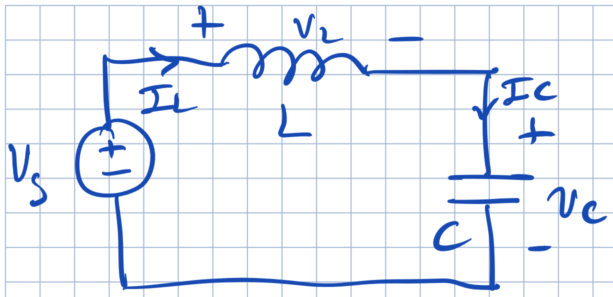
$$v_L(t) = \cancel{V_s(t)}^{\,0} - V_R(t) = -V_R(t)$$

$$= -R I_L(0)\, e^{-\frac{t}{\tau}}$$









$$I_c(t) = C\frac{dv_c(t)}{dt}$$

$$V_L(t) = L\frac{d}{dt}I_L(t)$$

$$I_L(t) = I_c(t) \quad (KCL)$$

$$V_s(t<0) = 1V$$

$$V_L(t) = V_s(t) - V_c(t)$$

$$V_s(t \geq 0) = 0V$$

For $t \geq 0$, $V_s(t) = 0V \rightarrow$

$$v_c(0) = 1V$$

$$v_L(t) = -v_c(t)$$

$$I_L(0) = 0A$$

Use $v_c(t)$ & $I_L(t)$ as state-variables.

$$L\frac{d}{dt}I_L(t) = -v_c(t) \quad (1)$$

$$I_L(t) = C\frac{dv_c(t)}{dt} \quad (2)$$

(1) $$\frac{d}{dt}I_L(t) = -\frac{1}{L}v_c(t)$$

(2) $$\frac{d}{dt}v_c(t) = \frac{1}{C}I_L(t)$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} I_L(t) \\ v_c(t) \end{bmatrix}}_{\frac{d}{dt}\vec{x}(t)} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_L(t) \\ v_c(t) \end{bmatrix}}_{\vec{x}(t)}$$

$\downarrow$ diagonalize A to solve.

Compute eigenvalues of A:

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & \frac{1}{L} \\ -\frac{1}{C} & \lambda \end{pmatrix} =$$

$$= \lambda^2 + \frac{1}{LC} = 0$$

$$\Downarrow$$

$$\boxed{\lambda_{1,2} = \pm j\sqrt{\frac{1}{LC}} \quad , \; j = \sqrt{-1}}$$

Assume: $L = 1H$, $C = 1F$ (large artificial values)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm j$$

want to find $\vec{v_1}$ & $\vec{v_2}$ s.t.

$$A\vec{v_1} = \lambda_1 \vec{v_1} \quad \& \quad A\vec{v_2} = \lambda_2 \vec{v_2}$$

Nullspace style

$\lambda_1 = j$

$$(A - \lambda_1 I)\cdot\vec{v_1} = \vec{0}$$

$$\begin{bmatrix} -j & -1 \\ 1 & -j \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v_1} = \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

$\lambda_2 = -j$

$$\vec{v_2} = \begin{bmatrix} 1 \\ ? \end{bmatrix}$$

Any multiple of eigenvector is also an eigenvector from eigenspace

$$A k\vec{v_2} = \lambda_2 k\vec{v_2}$$

so can normalize one element of $\vec{v_2}$.

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ x \end{bmatrix}}_{\vec{v_2}} = \underbrace{-j}_{\lambda_2}\underbrace{\begin{bmatrix} 1 \\ x \end{bmatrix}}_{\vec{v_2}}$$

$$V = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \leftarrow \text{eigenbasis}$$

$-x = -j \Rightarrow x = j$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{j}{2} \\ \frac{1}{2} & -\frac{j}{2} \end{bmatrix}$$

$$\vec{x}(t) = V \vec{\tilde{x}}$$

$$\begin{tikzcd} \vec{x}(t) \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \\ \vec{\tilde{x}} \arrow[u, bend right, "V"'] & \frac{d}{dt}\vec{\tilde{x}} \end{tikzcd}$$

$$= [\vec{v}_1 \dots \vec{v}_n] \begin{bmatrix} \tilde{x}_1(t) \\ \vdots \\ \tilde{x}_n(t) \end{bmatrix}$$

$$= \underbrace{\tilde{x}_1(t)}_{\text{coordinate}} \underbrace{\vec{v}_1}_{\text{basis}} + \dots + \tilde{x}_n \vec{v}_n$$

Found a way to express $\vec{x}$ in V-eigen basis with $\vec{\tilde{x}}$ coordinates.

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \vec{x}(t)$$

$$\Downarrow$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \underbrace{\begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}}_{V^{-1}AV} \vec{\tilde{x}}(t)$$

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \tilde{x}_1(0) e^{jt} \\ \tilde{x}_2(0) e^{-jt} \end{bmatrix}$$

need $\vec{\tilde{x}}(0)$

$I_L(0) = 0A$

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} \frac{1}{2} & \frac{j}{2} \\ \frac{1}{2} & -\frac{j}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{j}{2} \\ -\frac{j}{2} \end{bmatrix}$$

$v_c(0) = 1V$

$$\vec{\tilde{x}}(t) = \begin{bmatrix} \frac{j}{2} e^{jt} \\ -\frac{j}{2} e^{-jt} \end{bmatrix}$$

$$\vec{x}(t) = V\vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} \frac{j}{2} e^{jt} \\ -\frac{j}{2} e^{-jt} \end{bmatrix}$$

$$\begin{bmatrix} I_L(t) \\ v_c(t) \end{bmatrix} = \vec{x}(t) = \begin{bmatrix} \frac{j}{2} e^{jt} - \frac{j}{2} e^{-jt} \\ \frac{e^{jt} + e^{-jt}}{2} \end{bmatrix}$$

Euler formula (or Taylor expansion)

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$x_1(t) = \frac{j}{2} e^{jt} - \frac{j}{2} e^{-jt} =$$

$$= \frac{j}{2}\left(e^{jt} - e^{-jt}\right) = \frac{j}{2}\left(\cos t + j\sin t - (\cos(-t) + j\sin(-t))\right)$$

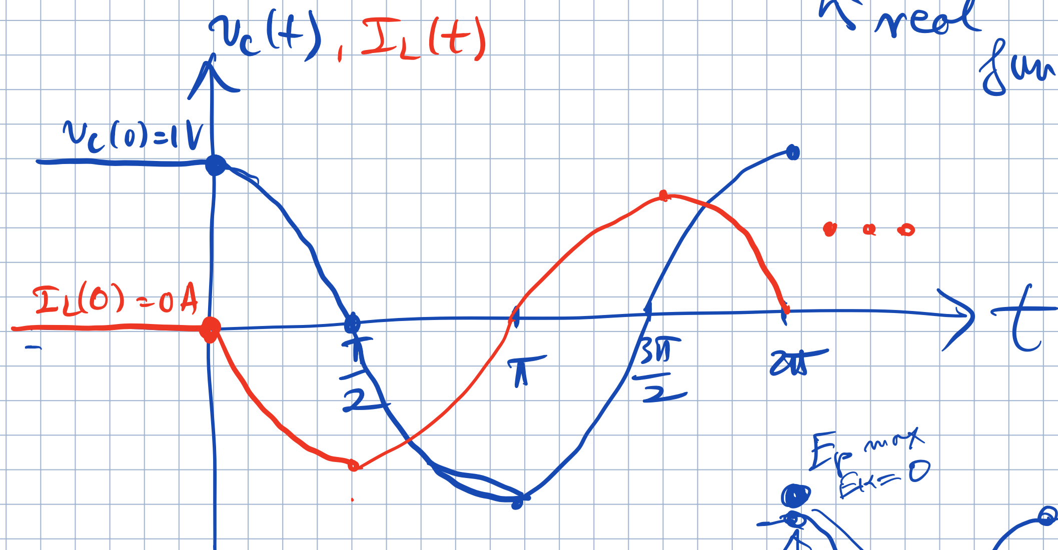
$$= \frac{j}{2}\left(\cos t + j\sin t - (\cos t - j\sin t)\right)$$

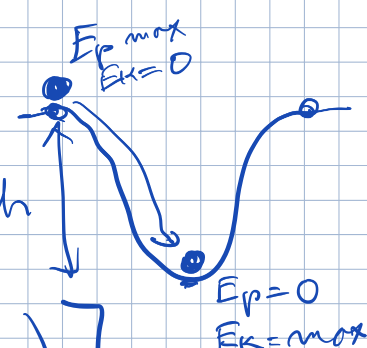
$$= \frac{j}{\cancel{2}} \cdot \cancel{2}j \sin t = -\sin t$$

$$x_2(t) = \frac{e^{jt} + e^{-jt}}{2} = \cos t$$

$$\Rightarrow \quad \vec{x}(t) = \begin{bmatrix} I_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

↑ real functions





Note: $\lambda_{1,2} = \pm j \sqrt{\frac{1}{2C}}$

$$\vec{x}(t) = \begin{bmatrix} I_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\sin\left(\frac{1}{\sqrt{2C}} t\right) \\ \cos\left(\frac{1}{\sqrt{2C}} t\right) \end{bmatrix}$$

Refresher:

For $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$, when $u(t) = K \cdot e^{st}$ & $s \neq \lambda$

$$x(t) = \underbrace{\left(x(0) - \frac{K}{s-\lambda}\right)e^{\lambda t}}_{} + \underbrace{\frac{K}{s-\lambda}e^{st}}_{}$$

Annoying – transient solution b/c of initial conditions

Nice – same form as input (steady-state soln)

Want the first part to disappear for $t \to \infty$.

If $\lambda < 0$, $e^{\lambda t} \xrightarrow[t \to \infty]{} 0$ and

$x(t) \to$ steady-state $\left(\frac{K}{s-\lambda}e^{st}\right)$

What about λ complex?

$$e^{\lambda t} = e^{(\lambda_r + j\lambda_i)t} = e^{\lambda_r t} \cdot e^{j\lambda_i t} =$$

$$= e^{\lambda_r t}\left(\cos(\lambda_i t) + j\sin(\lambda_i t)\right)$$

If $\lambda_r < 0$ $\quad e^{\lambda_r t} \xrightarrow[t \to \infty]{} 0$

$x(t) \to$ steady-state