Lecture 8

* Solving systems of diff. eqns with Phasors
* i.e. transforming systems of diff. eqns into sys. of lin. eqns.

$$(1) \qquad \frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t)$$

Let's consider the inputs of the form $\sim e^{st}$ and assert that the solutions are $\sim e^{st}$, valid for $s \neq \lambda$, $Re(\lambda) = \lambda_r < 0$ in steady-state.

$$\vec{u}(t) = \vec{\tilde{u}}\, e^{st}, \quad \vec{\tilde{u}} \text{ vector of constants}$$

Assert: $\vec{x}(t) = \vec{\tilde{x}}\, e^{st}$, $\vec{\tilde{x}}$ vector of constants

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}(\vec{\tilde{x}} e^{st}) = \vec{\tilde{x}} \frac{d}{dt} e^{st} = s\vec{\tilde{x}} e^{st}$$

From (1) $\quad s\vec{\tilde{x}}\cancel{e^{st}} = A \cdot \vec{\tilde{x}}\cancel{e^{st}} + \vec{\tilde{u}}\cancel{e^{st}}$

$$s\vec{\tilde{x}} = A\vec{\tilde{x}} + \vec{\tilde{u}}$$

$$(2) \qquad \boxed{(sI - A)\vec{\tilde{x}} = \vec{\tilde{u}}}$$

System of lin. equations.

$$\boxed{\vec{\tilde{x}} = (sI - A)^{-1} \cdot \vec{\tilde{u}}}$$

[Remember $s \neq \lambda$
$sI - A$ has no nullspace & is therefore

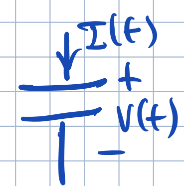
$$\vec{x}(t) = \hat{\vec{x}} \cdot e^{st}$$

invertible

$$\boxed{\vec{x}(t) = (sI - A)^{-1}\, \hat{\vec{u}} \cdot e^{st}}$$

solution to (1) for $\vec{u}(t) = \hat{\vec{u}} e^{st}$

Can we use this for ckts directly?

C 

$$I(t) = C \frac{d}{dt} V(t)$$

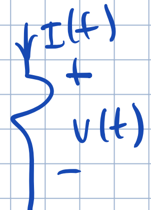
$$\left.\begin{aligned} I(t) &= \hat{I} e^{st} \\ V(t) &= \hat{V} e^{st} \end{aligned}\right\} \frac{V(t)}{I(t)} = \frac{\hat{V}}{\hat{I}}$$

$$I(t) = \hat{I} e^{st} = C \frac{d}{dt}(\hat{V} e^{st}) = C\hat{V} \frac{d}{dt}(e^{st})$$

$$\hat{I} \cancel{e^{st}} = sC\hat{V} \cancel{e^{st}} \qquad = sC\hat{V} e^{st}$$

$$\boxed{\frac{\hat{V}}{\hat{I}} = \frac{1}{sC}}$$

(capacitor s-impedance)

R 

$$V(t) = \hat{V} e^{st}$$

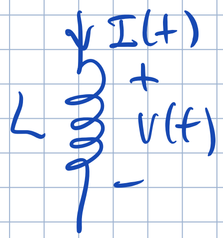
$$I(t) = \hat{I} e^{st}$$

$$V(t) = R \cdot I(t)$$

$$\hat{V} \cancel{e^{st}} = R \hat{I} \cancel{e^{st}}$$

$$\boxed{\frac{\hat{V}}{\hat{I}} = R}$$

(resistor s-impedance)



$$V(t) = L \frac{d}{dt} I(t)$$

$$V(t) = \hat{V} e^{st}$$

$$I(t) = \hat{I} e^{st}$$

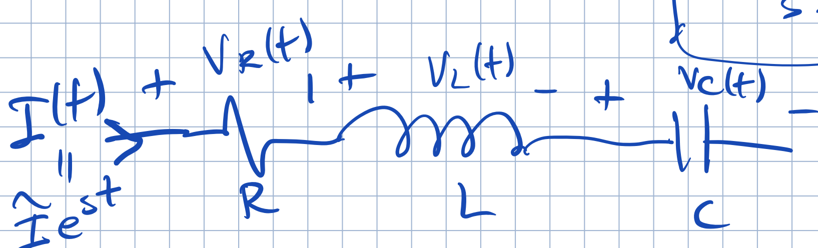
$$\hat{V} e^{st} = L \frac{d}{dt}(\hat{I} e^{st}) = L\hat{I} \frac{d}{dt}(e^{st})$$

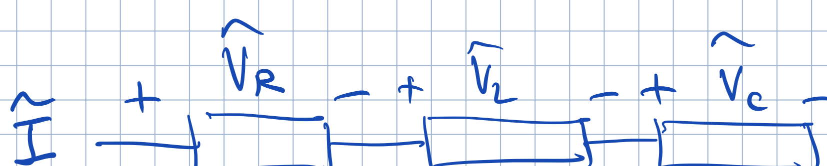
$$= sL\hat{I} e^{st}$$

$$\hat{V} \cancel{e^{st}} = sL\hat{I} \cancel{e^{st}}$$

$$\hat{V} = sL\hat{I} \Rightarrow \boxed{\frac{\hat{V}}{\hat{I}} = sL}$$

inductor
s-impedance





$$Z_R(s) = R \qquad Z_L(s) = sL \qquad Z_C(s) = \frac{1}{sC}$$

$$\hat{V}_R = \hat{I} \cdot Z_R, \quad \hat{V}_L = \hat{I} Z_L, \quad \hat{V}_C = \hat{I} \cdot Z_C$$

$$V_R(t) = \hat{V}_R e^{st}, \quad \ldots$$

For sinusoidal inputs:

$$u(t) = U\cos(\omega t + \phi) = U\,\frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2}$$

$$u(t) = \underbrace{\underbrace{\frac{U\cdot e^{j\phi}}{2}}_{\tilde{u}}\cdot\underbrace{e^{j\omega t}}_{e^{s_1 t}}}_{s_1 = j\omega} + \underbrace{\underbrace{\frac{U e^{-j\phi}}{2}}_{\bar{\tilde{u}}}\cdot\underbrace{e^{-j\omega t}}_{e^{s_2 t}}}_{s_2 = -j\omega}$$

always complex conjugate b/c $u(t)$ is real

$$u(t) = \tilde{u}\, e^{j\omega t} + \bar{\tilde{u}}\, e^{-j\omega t}$$

Use superposition to solve for the first & second component.

For $s_1 = j\omega$: $\quad M = s_1 I - A = j\omega I - A$

$$M\vec{\tilde{x}} = \vec{\tilde{u}}$$

$$\vec{\tilde{x}} = M^{-1}\vec{\tilde{u}}$$

element voltages & currents $\rightarrow$ $\begin{bmatrix}\vec{\tilde{I}}\\ \vec{\tilde{V}}\end{bmatrix} = M^{-1}\,\tilde{u}$ $\leftarrow$ independent sources

M^{-1}: $\uparrow$ circuit topology

For $s_2 = -j\omega$ $\qquad \bar{M} = s_2 I - A = -j\omega I - A$

$$\bar{M}\begin{bmatrix} \vec{\tilde{\hat{I}}} \\ \vec{\tilde{\hat{V}}} \end{bmatrix} = \vec{\tilde{\hat{u}}} \qquad \overline{M\vec{x}} = \bar{M}\,\bar{\vec{x}}$$

$$\overline{\begin{bmatrix} \vec{\hat{I}} \\ \vec{\hat{V}} \end{bmatrix}} = \begin{bmatrix} \vec{\tilde{\hat{I}}} \\ \vec{\tilde{\hat{V}}} \end{bmatrix} = \bar{M}^{-1} \cdot \vec{\tilde{\hat{u}}} = \overline{M^{-1}\vec{\hat{u}}}$$

$$\begin{bmatrix} \vec{\tilde{\hat{I}}} \\ \vec{\tilde{\hat{V}}} \end{bmatrix} = M^{-1}\vec{\hat{u}}$$

all solutions:

$$\vec{V}(t) = \vec{\hat{V}} e^{j\omega t} + \vec{\tilde{\hat{V}}} e^{j\omega t}$$

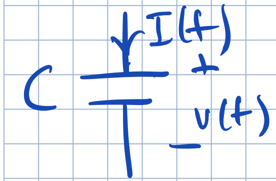
$$\vec{I}(t) = \vec{\hat{I}} e^{j\omega t} + \vec{\tilde{\hat{I}}} e^{j\omega t}$$

so only need to find one (i.e. $\vec{\hat{V}}, \vec{\hat{I}}$)
to compute $\vec{V}(t)$ & $\vec{I}(t)$

For sinusoidal inputs:
$\hat{V}$ and $\hat{I}$ are phasors (functions of $s = j\omega$)

s-impedances are called impedances

Phasors turn 16B problems into 16A problems!



$$v(t) = V_0 \cos(\omega t + \phi)$$

$$v(t) = \underbrace{\frac{V_0}{2} e^{j\phi}}_{\hat{V}} e^{j\omega t} + \underbrace{\frac{V_0}{2} e^{-j\phi}}_{\bar{\hat{V}}} e^{-j\omega t}$$

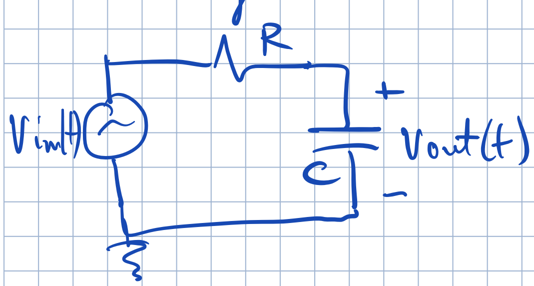
$$I(t) = C \frac{d}{dt} v(t) = C \frac{d}{dt}\left(\hat{V} e^{j\omega t} + \bar{\hat{V}} e^{-j\omega t}\right)$$

$$= \underbrace{j\omega C \hat{V}}_{\hat{I}} e^{j\omega t} + \underbrace{(-j\omega) C \bar{\hat{V}}}_{\bar{\hat{I}}} e^{-j\omega t}$$

$$I(t) = \hat{I} e^{j\omega t} + \bar{\hat{I}} e^{-j\omega t}$$

$$\hat{I} = j\omega C \hat{V} \Rightarrow \boxed{Z_C(s = j\omega) = \frac{1}{j\omega C}}$$

Example 1 : RC circuit



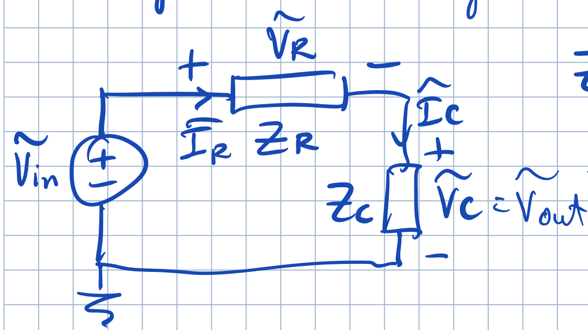
$$V_{in}(t) = V_{in} \cos(\omega t + \phi)$$

$$V_{out}(t) = ?$$

step 1: Write the sources as exponentials

$$V_{in}(t) = \underbrace{\frac{V_{in}}{2} e^{+j\phi}}_{\hat{V}_{in} \text{ phasor}} e^{j\omega t} + \underbrace{\frac{V_{in}}{2} e^{-j\phi}}_{\bar{\hat{V}}_{in}} e^{-j\omega t}$$

step 2: Transform the circuit to phasor domain



$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C}$$

step 3: Write down the circuit equations

$$\hat{V}_R = Z_R \hat{I}_R \,, \qquad \hat{V}_C = \hat{I}_C \cdot Z_C$$

KCL: $\hat{I}_R = \hat{I}_C$

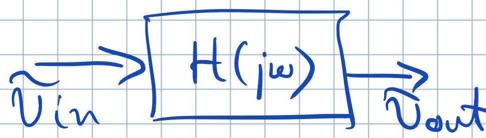
step 4: Solve the circuit

$$\hat{V}_C = \frac{Z_C}{Z_C + Z_R} \cdot \hat{V}_{in} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} \hat{V}_{in}$$

$$\hat{V}_C = \hat{V}_{out} = \frac{1}{1 + j\omega RC} \hat{V}_{in}$$

Note

$$\hat{V}_{out}(j\omega) = \frac{1}{1 + j\omega RC} \hat{V}_{in}(j\omega)$$



$$H(j\omega) = \frac{\hat{V}_{out}(j\omega)}{\hat{V}_{in}(j\omega)}$$

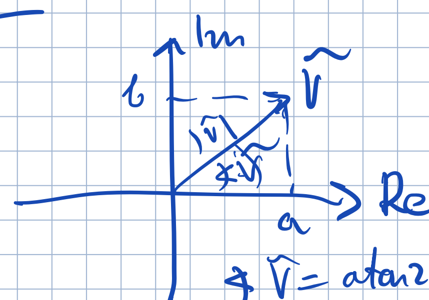
For our LP example: transfer function

$$H_{LP}(j\omega) = \frac{1}{1 + j\omega RC} = \frac{1 - j\omega RC}{1 + (\omega RC)^2}$$

$$|H(j\omega)| = \frac{1}{|1 + j\omega RC|} = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$\measuredangle H(j\omega) = -\operatorname{atan2}(\omega RC, 1)$$

$$\hat{V}_{out} = H(j\omega) \cdot \hat{V}_{in}$$



$$\measuredangle \hat{V} = \operatorname{atan2}(b, a)$$

$$|\hat{V}| = \sqrt{a^2 + b^2}$$

$$\hat{V}_{in} = |\hat{V}_{in}| \, e^{j \measuredangle \hat{V}_{in}}$$

$$\hat{V}_{out} = |\hat{V}_{out}| \, e^{j \measuredangle \hat{V}_{out}}$$

$$H(j\omega) = |H(j\omega)| \, e^{j \measuredangle H(j\omega)}$$

$$\hat{V}_{out} = |\hat{V}_{out}| \, e^{j \measuredangle \hat{V}_{out}} = |\hat{V}_{in}| \, e^{j \measuredangle \hat{V}_{in}} \cdot |H(j\omega)| \, e^{j \measuredangle H(j\omega)}$$

$$|\hat{V}_{out}| \, e^{j \measuredangle \hat{V}_{out}} = |\hat{V}_{in}| \cdot |H(j\omega)| \cdot e^{j(\measuredangle \hat{V}_{in} + \measuredangle H(j\omega))}$$

$$|\hat{V}_{out}| = |H(j\omega)| \, |\hat{V}_{in}|$$

$$\measuredangle \hat{V}_{out} = \measuredangle H(j\omega) + \measuredangle \hat{V}_{in}$$

$$|H_{LP}(j\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}} \rightarrow 1 \text{ as } \omega \rightarrow 0$$

i.e. $\omega << \frac{1}{RC}$

$\rightarrow 0$ as $\omega \rightarrow \infty$

i.e a low-pass filter

$\omega >> \frac{1}{RC}$

step 5: Convert to time domain.

$$V_{out}(t) = \hat{V}_{out} e^{j\omega t} + \bar{\hat{V}}_{out} e^{-j\omega t}$$

$$= |\hat{V}_{out}| e^{j\measuredangle \hat{V}_{out}} \cdot e^{j\omega t} + |\hat{V}_{out}| e^{-j\measuredangle \hat{V}_{out}} \cdot e^{-j\omega t}$$

$$= |\hat{V}_{out}| e^{j(\omega t + \measuredangle \hat{V}_{out})} + |\hat{V}_{out}| e^{-j(\omega t + \measuredangle \hat{V}_{out})}$$

$$= 2 |\hat{V}_{out}| \cos(\omega t + \measuredangle \hat{V}_{out})$$

similarly: $V_{in}(t) = \underbrace{2|\hat{V}_{in}|}_{V_{in}} \cos(\omega t + \underbrace{\measuredangle \hat{V}_{in}}_{\phi})$

$$V_{out}(t) = 2 \cdot |H_{LP}(j\omega)| \cdot |\hat{V}_{in}| \cdot \cos(\omega t + \measuredangle H_{LP}(j\omega) + \measuredangle \hat{V}_{in})$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}} \cdot V_{in} \cos(\omega t + \underbrace{\measuredangle H_{LP}(j\omega)}_{-\mathrm{atan2}(\omega RC, 1)} + \underbrace{\measuredangle V_{in}}_{\phi})$$

what is the phase?

$$\omega \cdot \underset{\uparrow \text{delay}}{t_d} = \measuredangle H(j\omega) \quad , \quad t_d = \frac{\measuredangle H(j\omega)}{\omega}$$