Lecture 9

* Filters
  * Low-pass & high-pass examples
  * Transfer functions
  * Bode plot approximations
* Cascading filters

Objective: To find a steady-state solution of the system in response to sinusoidal inputs. ($\lambda_r < 0$)

Sinusoidal inputs allow us to transform the system of diff. eqns into a system of lin. eqns. (Phasor domain)

Phasor analysis:

Step 1) Write sources (inputs) as exponentials:

$$V_{si}(t) = \hat{V}_{si} e^{j\omega t} + \bar{\hat{V}}_{si} e^{-j\omega t}$$

$$I_{si}(t) = \hat{I}_{si} e^{j\omega t} + \bar{\hat{I}}_{si} e^{-j\omega t}$$

$\hat{V}_{si}$ & $\hat{I}_{si}$ are phasors

Step 2) Transform the circuit into the phasor domain:

(s - impedances $s = j\omega$) $\dfrac{\hat{V}_{el}}{\hat{I}_{el}} = Z_{el}$

$$\frac{\hat{V}_R}{\hat{I}_R} = Z_R = R \quad , \quad \frac{\hat{V}_C}{\hat{I}_C} = Z_C(j\omega) = \frac{1}{j\omega C} \quad , \quad \frac{\hat{V}_L}{\hat{I}_L} = Z_L(j\omega) = j\omega L$$

Generalized Ohm's Law.

Step 3) Cast branch & element equations in phasor domain:

KCL: $\sum_{\substack{i \text{ into} \\ \text{the node}}} \hat{I}_i = 0$ Ohm's law: $\hat{V}_i = Z_i \hat{I}_i$

NVA: $\sum_i \frac{\hat{V}_j - \hat{V}_k}{Z_{jk}} = 0$, KVL holds as well.

Step 4) Solve for unknown variables:

$\hat{V}_R, \hat{I}_R$, $\hat{V}_L, \hat{I}_L$, $\hat{V}_C, \hat{I}_C$

(in general $\hat{V}el_i, \hat{I}el_i$)

Step 5) Transform the phasor solutions from step 4 into the time domain

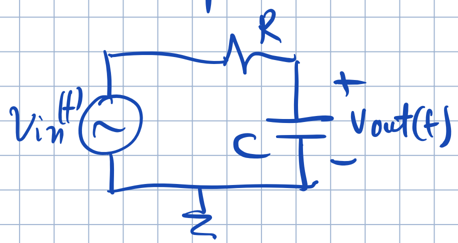
$$Vel(t) = \hat{V}el\, e^{j\omega t} + \bar{\hat{V}}el\, e^{-j\omega t}$$

$$Iel(t) = \hat{I}el\, e^{j\omega t} + \bar{\hat{I}}el\, e^{-j\omega t}$$

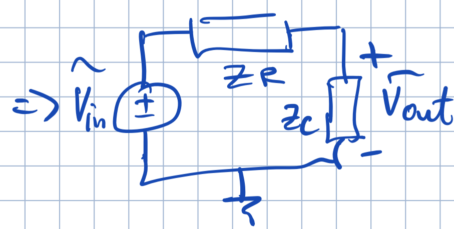
$$Vel(t) = 2|\hat{V}el| \cos(\omega t + \measuredangle \hat{V}el)$$

$$Iel(t) = 2|\hat{I}el| \cos(\omega t + \measuredangle \hat{I}el)$$

Example 1 continued (Low-pass filter):



$\Rightarrow$



$$\tilde{V}_{out} = \frac{Z_C}{Z_C + Z_R} \tilde{V}_{in}$$

$$\underbrace{\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = H_{LP}(j\omega)}_{\text{transfer function}} = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC}$$

$$\omega_0 = \frac{1}{RC}$$ we call ω_0 "cut-off freq"

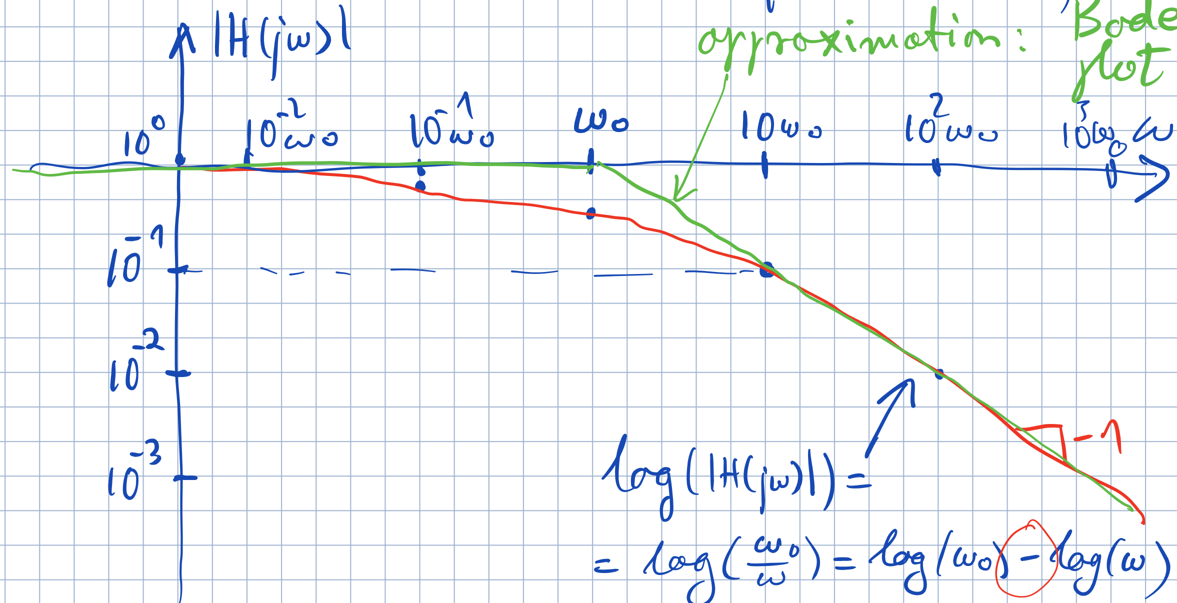
$$\boxed{H_{LP}(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}}}$$

$$|H_{LP}(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \xrightarrow{\omega \gg \omega_0} 0$$

$$\xrightarrow{\omega \ll \omega_0} 1$$

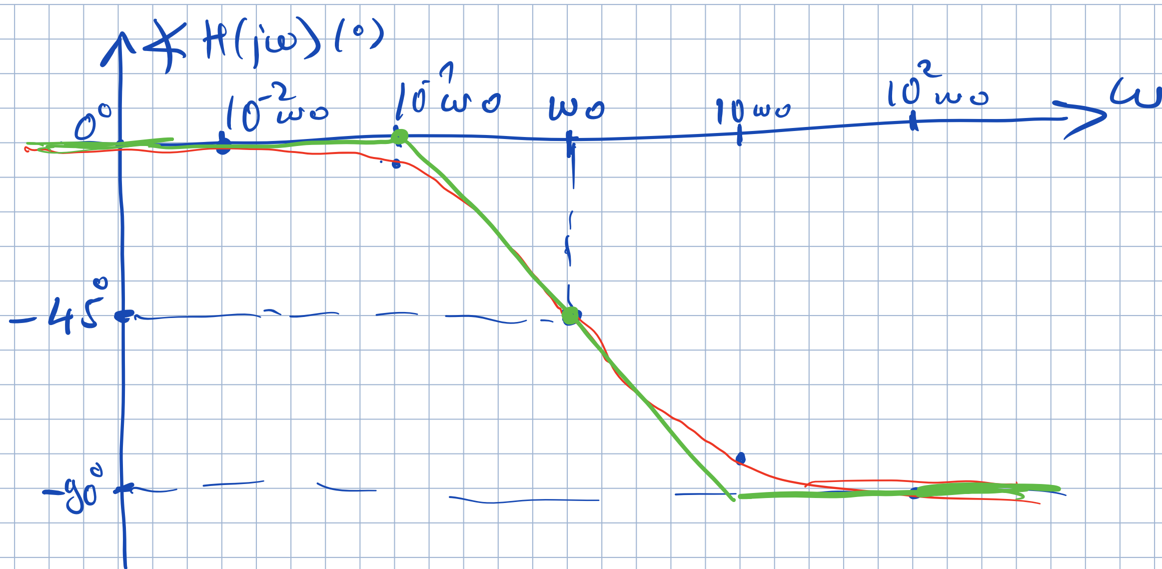
$$\measuredangle H_{LP}(j\omega) = -\operatorname{atan}\left(\frac{\omega}{\omega_0}, 1\right)$$

ω	$H(j\omega)$	$\lvert H(j\omega)\rvert$	$\measuredangle H(j\omega)$
$\omega \ll \omega_0$	$\simeq 1$	1	$\sim 0°$
$0.1\,\omega_0$	$\frac{1}{1+j0.1}$	0.995	$-6°$
ω_0	$\frac{1}{1+j}$	$\frac{1}{\sqrt{2}} \approx 0.71$	$-45°$
$10\,\omega_0$	$\frac{1}{1+j10}$	0.1	$-84°$
$\omega \gg \omega_0$	$-j\frac{\omega_0}{\omega}$	$\frac{\omega_0}{\omega}$	$-90°$



$$\log(|H(j\omega)|) =$$
$$= \log\left(\frac{\omega_0}{\omega}\right) = \log(\omega_0) - \log(\omega)$$

"slope" $= -1$



$$V_{out}(t) = \hat{V}_{out} e^{j\omega t} + \bar{\hat{V}}_{out} e^{-j\omega t}$$

$$= |\hat{V}_{out}| e^{j\angle \hat{V}_{out}} \cdot e^{+j\omega t} + |\hat{V}_{out}| e^{-j\angle \hat{V}_{out}} \cdot e^{-j\omega t}$$

$$= |\hat{V}_{out}| \left(\underbrace{e^{j(\omega t + \angle \hat{V}_{out})} + e^{-j(\omega t + \angle \hat{V}_{out})}}_{\text{Euler: } 2\cos(\omega t + \angle \hat{V}_{out})} \right)$$

$$= 2|\hat{V}_{out}| \cos(\omega t + \angle \hat{V}_{out})$$

$$\hat{V}_{out} = H_{LP}(j\omega) \cdot \hat{V}_{in}$$

$$|\hat{V}_{out}| \cdot e^{j\angle \hat{V}_{out}} = |H_{LP}(j\omega)| \cdot e^{j\angle H_{LP}(j\omega)} \cdot |\hat{V}_{in}| e^{j\angle \hat{V}_{in}}$$

$$|\hat{V}_{out}| e^{j\angle \hat{V}_{out}} = |H_{LP}(j\omega)| \cdot |\hat{V}_{in}| \cdot e^{j(\angle \hat{V}_{in} + \angle H_{LP}(j\omega))}$$

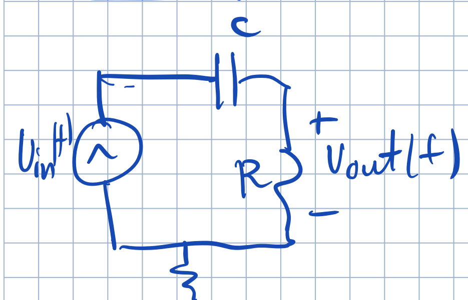
$$\boxed{\begin{aligned} |\hat{V}_{out}| &= |H_{LP}(j\omega)|\,|\hat{V}_{in}| \\ \angle \hat{V}_{out} &= \angle \hat{V}_{in} + \angle H_{LP}(j\omega) \end{aligned}}$$

$$V_{out}(t) = 2 \cdot |H_{LP}(j\omega)| \cdot |\hat{V}_{in}| \cos(\omega t + \measuredangle \hat{V}_{in} + \measuredangle H_{LP}(j\omega))$$

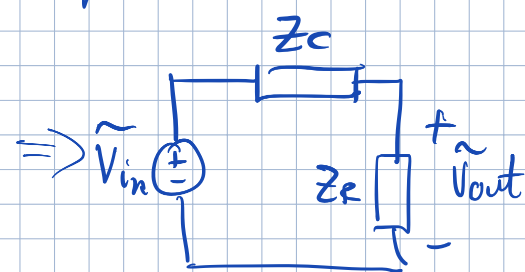
our input $v_{in}(t) = V_{in} \cos(\omega t + \phi)$

$$\boxed{V_{out}(t) = |H_{LP}(j\omega)| \cdot V_{in} \cos(\omega t + \phi + \measuredangle H_{LP}(j\omega))}$$

Example 2: (High-pass filter)



time-domain



$$\tilde{V}_{out} = \frac{Z_R}{Z_R + Z_C} \hat{V}_{in}$$

$$H_{HP}(j\omega) = \frac{\tilde{V}_{out}}{\hat{V}_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + \frac{1}{j\omega C}}$$

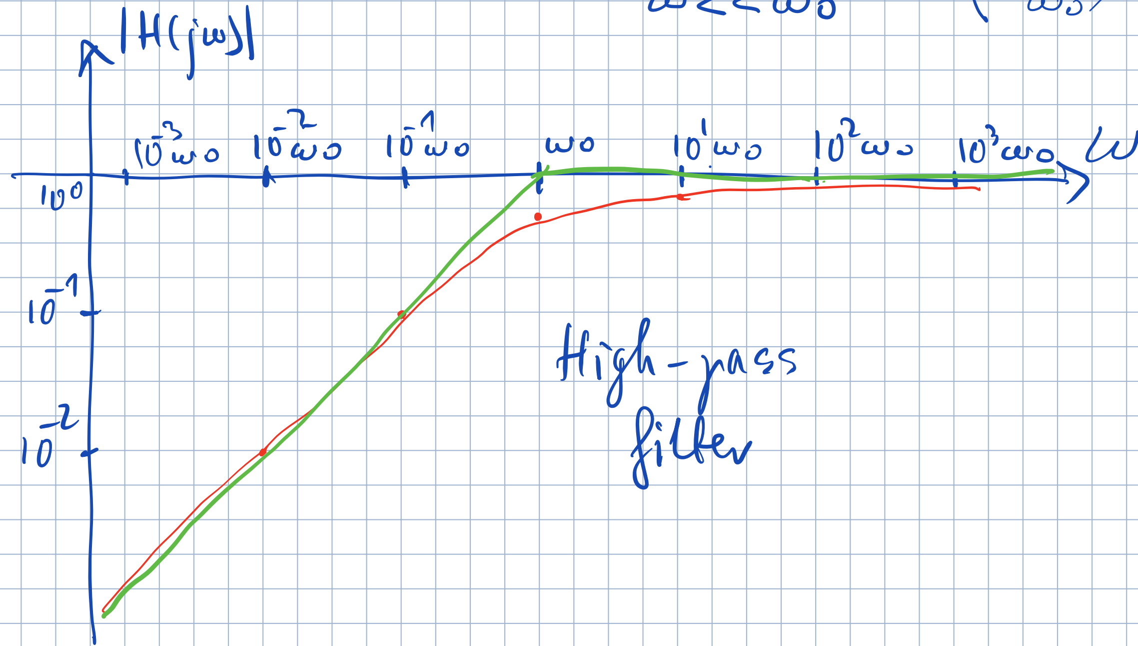
$$H_{HP}(j\omega) = \frac{j\omega RC}{1 + j\omega RC} \quad , \quad \omega_0 = \frac{1}{RC}$$

"cut-off freq"

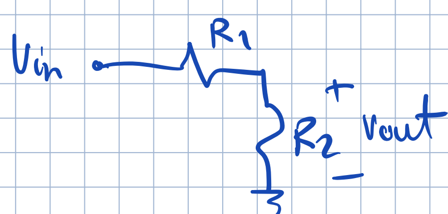
$$H_{HP}(j\omega) = \frac{j\frac{\omega}{\omega_0}}{1 + j\frac{\omega}{\omega_0}} = \frac{1}{1 - j\frac{\omega_0}{\omega}}$$

$$|H_{HP}(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega_0}{\omega}\right)^2}} \xrightarrow{\omega \gg \omega_0} 1$$

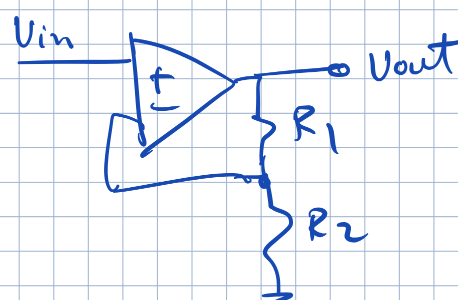
$$\xrightarrow[\omega \ll \omega_0]{} 0 \quad \left(\frac{\omega}{\omega_0}\right)$$



In 16 A:



$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1+R_2}$$

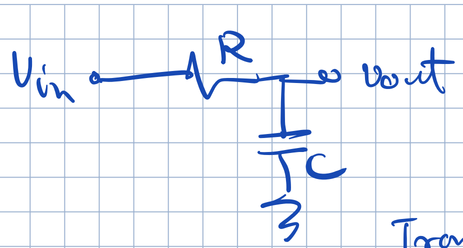


$$\frac{V_{out}}{V_{in}} = 1 + \frac{R_1}{R_2}$$

"Operators"

Transfer functions

In 16 B:



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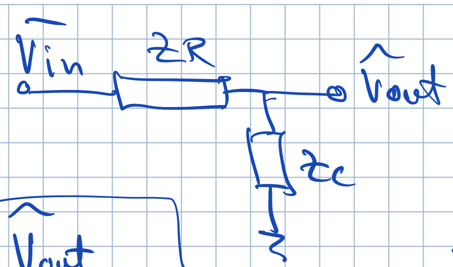
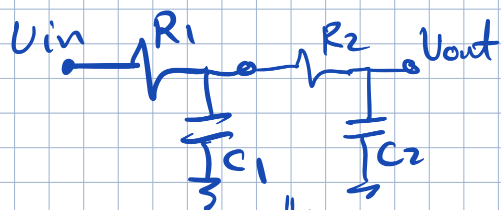


Transfer function $\boxed{\dfrac{\hat{V}_{out}}{\hat{V}_{in}} = H(j\omega)} = \dfrac{Z_C}{Z_C + Z_R}$

How do we cascade circuits in a tractable way to build more complex functions?

Circuit blocks should not "load" (i.e. take current from) each other in order to preserve the transfer function.



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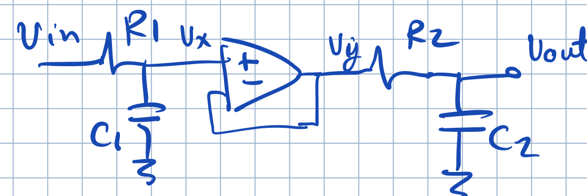
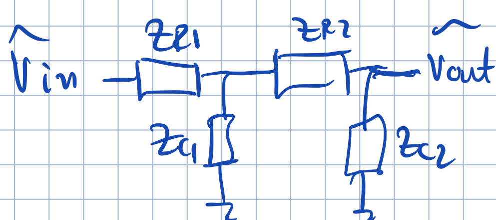


⇓ Phasor domain



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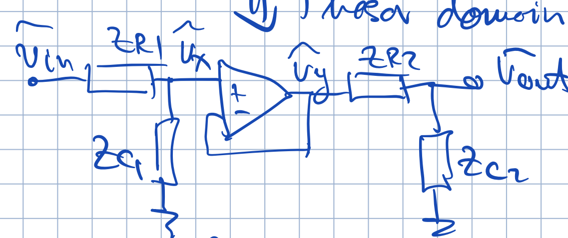

⇓

Solve $H(j\omega) = \dfrac{\hat{V}_{out}}{\hat{V}_{in}}$

Need to calculate the soln.

$$H_1(j\omega) = \frac{\hat{V}_x}{\hat{V}_{in}}, \quad H_{buf}(j\omega) = \frac{\hat{V}_y}{\hat{V}_x}$$

$$H_2(j\omega) = \frac{\hat{V}_{out}}{\hat{V}_y}$$

$$H(j\omega) = \frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{\hat{V}_{out}}{\hat{V}_y} \cdot \frac{\hat{V}_y}{\hat{V}_x} \cdot \frac{\hat{V}_x}{\hat{V}_{in}}$$

$$= H_2(j\omega) \cdot H_{buf}(j\omega) \cdot H_1(j\omega)$$

$$H_1(j\omega) = \frac{Z_{C1}}{Z_{C1} + Z_{R1}}$$

voltage div.

$$H_{buf}(j\omega) = 1$$

$$\hat{V}_y = \hat{V}_x$$

$$H_2(j\omega) = \frac{Z_{C2}}{Z_{C2} + Z_{R2}}$$

voltage - div.

$$H_1(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_{01}}} \quad , \quad H_2(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_{02}}}$$

$$\omega_{01} = \frac{1}{R_1 C_1} \quad , \quad \omega_{02} = \frac{1}{R_2 C_2}$$

$$H(j\omega) = H_1(j\omega) \cdot \overset{1}{\cancel{H_{buf}(j\omega)}} \cdot H_2(j\omega)$$

$$= \frac{1}{1 + j\frac{\omega}{\omega_{01}}} \cdot \frac{1}{1 + j\frac{\omega}{\omega_{02}}}$$

In general :

$$\tilde{V}_{in} \to \underbrace{\boxed{H_1(j\omega)} \to \cdots \to \boxed{H_n(j\omega)}}_{H(j\omega)} \to$$

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = H_1(j\omega)\cdot \ldots \cdot H_n(j\omega)$$

$$H_i(j\omega) = |H_i(j\omega)| \cdot e^{j\angle H_i(j\omega)}$$

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = H(j\omega) = \underbrace{|H_1(j\omega)|\cdots|H_n(j\omega)|}_{|H(j\omega)|}\, e^{j(\underbrace{\angle H_1(j\omega)+\ldots+\angle H_n(j\omega)}_{\angle H(j\omega)})}$$

Time domain:

$$v_{out}(t) = |H(j\omega)|\, 2|\hat{V}_{in}| \cos(\omega t + \angle \hat{V}_{in} + \angle H(j\omega))$$

for $v_{in}(t) = V_{in}\cos(\omega t+\phi)$

$$V_{out}(t) = |H(j\omega)|\, V_{in} \cos(\omega t + \phi + \angle H(j\omega)$$