Midterm 1: V1

1. **Solar Panel (6 pts)**

A particular solar panel's IV curves (labeled by irradiance) are given by:

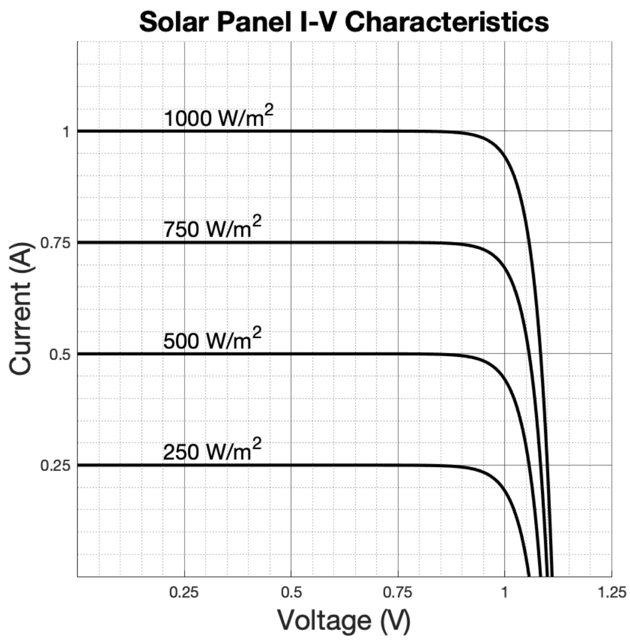
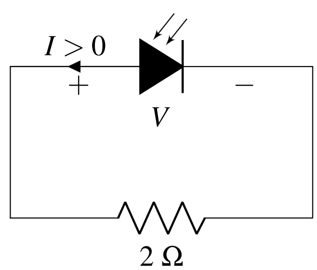


Figure 1: Solar Panel IV Curve

Your solar panel is connected to a load with resistance $R = 2\ \Omega$. The circuit diagram is given below, where the solar panel is depicted as a single pn-junction diode. Remember that current appears to flow "backward" out of the solar cell diode because it is *producing* power, rather than dissipating it.



(a) (3 pts) **Draw the resistor's IV curve on the plot which includes the solar panel IV curves above.** The intersection between your resistance curve and a particular IV curve tells you where the solar panel will end up operating given a particular irradiance.

Solution: We know that the current-voltage relationship for resistors is $I = \frac{V}{R}$.

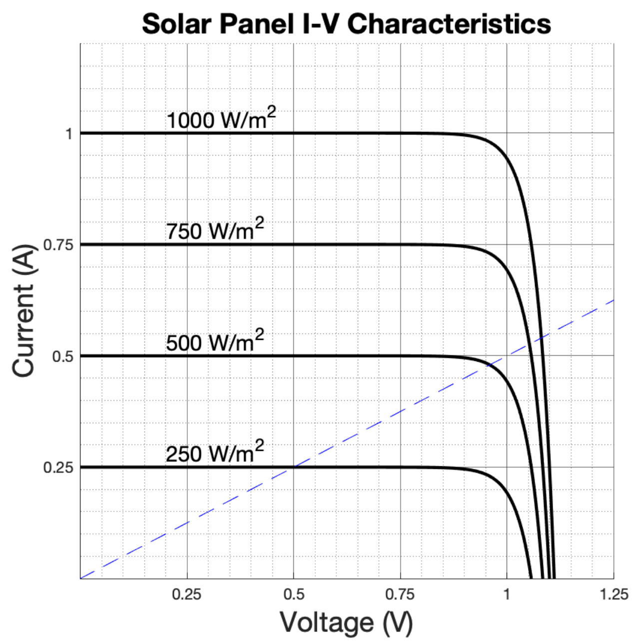


Figure 2: Solar Panel IV Curve with Resistor

(b) (3 pts) **If the irradiance were 250 W/m^2, what would the power dissipated across the resistor be?**

Solution: The 2Ω resistor intersects the 250 W/m^2 curve at $V = 0.5$ V and $I = 0.25$ A. The power dissipated is:

$$P = (0.5\ V)(0.25\ A) = 0.125W$$

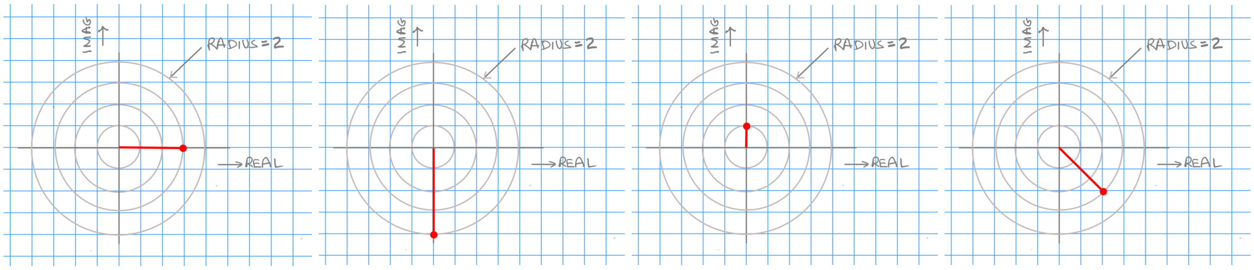
2. Phasors (30 pts)

(a) (3 pts) Consider a resistor ($R = 1.5\Omega$), a capacitor ($C = 1$F), and an inductor ($L = 1$H) connected in series. **Give expressions for the impedances of Z_R, Z_C, Z_L for each of these elements as a function of the angular frequency ω.**

Solution: The impedances are as follows: $Z_R = R = 1.5$, $Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega} = -\frac{j}{\omega}$ and $L = j\omega L = j\omega$.

(b) (3 pts) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = \frac{1}{2}$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.

Solution: Substituting for $\omega = \frac{1}{2}$ in the above answers, we get, $Z_R = 1.5$, $Z_C = -2j$ and $Z_L = 0.5j$. Since the elements are in series, $Z_{total} = Z_L + Z_C + Z_R = 1.5 - 1.5j$. Following are the plots:

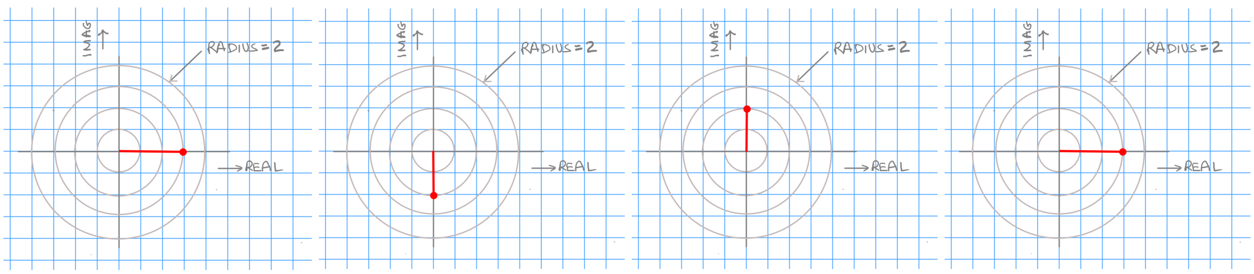


a $Z_R(@\omega = 0.5)$ b $Z_C(@\omega = 0.5)$ c $Z_L(@\omega = 0.5)$ d $Z_{total}(@\omega = 0.5)$

Figure 3: Impedances at $\omega = 0.5$.

(c) (3 pts) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 1$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.

Solution: Following the same method as last time, with $\omega = 1$, $Z_R = 1.5$, $Z_C = -j$, $Z_L = j$ and $Z_{total} = 1.5$

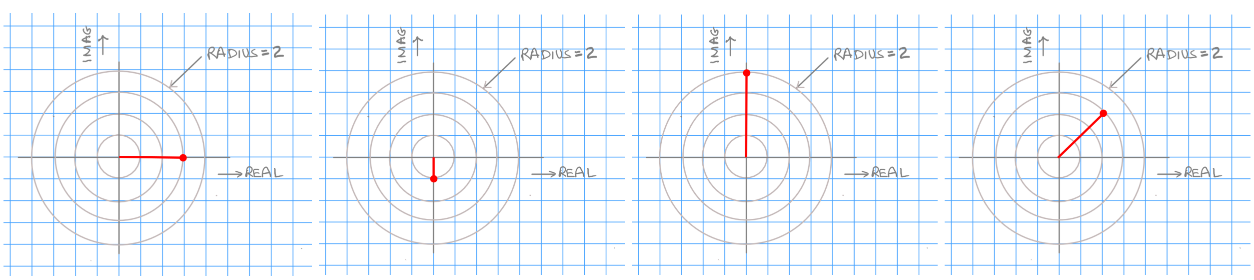


a $Z_R(@\omega = 1)$ b $Z_C(@\omega = 1)$ c $Z_L(@\omega = 1)$ d $Z_{total}(@\omega = 1)$

Figure 4: Impedances at $\omega = 1$.

(d) (3 pts) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 2$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Give the magnitude and phase of Z_{total}.** A logically sound graphical argument is sufficient justification.

Solution: Again, following the same method as last time, with $\omega = 1$, $Z_R = 1.5$, $Z_C = -0.5j$, $Z_L = 2j$ and $Z_{total} = 1.5 + 1.5j$



a $Z_R(@\omega = 2)$ b $Z_C(@\omega = 2)$ c $Z_L(@\omega = 2)$ d $Z_{total}(@\omega = 2)$

Figure 5: Impedances at $\omega = 2$.

(e) (3 pts) **For the previous series combination of RLC elements, what is the "natural frequency" ω_n where the series impedance is purely real?**

Solution: From our above answers, clearly the natural frequency, $\omega_n = 1$ rad/s

(f) (15 pts) Suppose that we have the two-dimensional system of differential equations expressed in matrix/vector form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{1}$$

where for this problem, we assume that $u(t)$ has a phasor representation $\widetilde{U}$. In other words, $u(t) = \widetilde{U}e^{+j\omega t} + \overline{\widetilde{U}}e^{-j\omega t}$. Suppose further that all the eigenvalues of A are such that any impact of an initial condition has completely died out by now. (i.e. the system is in steady-state.)

Assume that the vector solution $\vec{x}(t)$ to the system of differential equations (**??**) can also be written in phasor form as

$$\vec{x}(t) = \widetilde{\vec{X}}e^{+j\omega t} + \overline{\widetilde{\vec{X}}}e^{-j\omega t}. \tag{2}$$

Derive an expression for $\widetilde{\vec{X}}$ involving $A, \vec{b}$, $j\omega$, $\widetilde{U}$, and the identity matrix I.

*(HINT: Plug (**??**) into (**??**) and simplify, using the rules of differentiation and grouping terms by which exponential $e^{\pm j\omega t}$ they multiply.)*

Solution: As the hint suggests, plugging back (**??**) into (**??**) we get the following:

$$\frac{d}{dt}(\widetilde{\vec{X}}e^{j\omega t} + \overline{\widetilde{\vec{X}}}e^{-j\omega t}) = A(\widetilde{\vec{X}}e^{j\omega t} + \overline{\widetilde{\vec{X}}}e^{-j\omega t}) + \vec{b}\widetilde{U}e^{+j\omega t} + \overline{\widetilde{U}}e^{-j\omega t} \tag{3}$$

$$(j\omega\widetilde{\vec{X}}e^{j\omega t} - j\omega\overline{\widetilde{\vec{X}}}e^{-j\omega t}) = (A\widetilde{\vec{X}} + \vec{b}\widetilde{U})e^{j\omega t} + (A\overline{\widetilde{\vec{X}}} + \vec{b}\overline{\widetilde{U}})e^{-j\omega t} \tag{4}$$

$$\tag{5}$$

Note that $\widetilde{\vec{X}}$ and $\widetilde{U}$ do not depend on time since they are phasors. Next, we can group the coeffecients with the same exponenetial terms,

$$j\omega\widetilde{\vec{X}} = A\widetilde{\vec{X}} + \vec{b}\widetilde{U} \tag{6}$$

$$-j\omega\overline{\widetilde{\vec{X}}} = A\overline{\widetilde{\vec{X}}} + \vec{b}\overline{\widetilde{U}} \tag{7}$$

$$\Rightarrow \overline{(j\omega)}\overline{\widetilde{\vec{X}}} = \overline{(A\widetilde{\vec{X}} + \vec{b}\widetilde{U})} \tag{8}$$

$$\Rightarrow (j\omega)\widetilde{\vec{X}} = (A\widetilde{\vec{X}} + \vec{b}\widetilde{U}) \tag{9}$$

We see that equations (**??**) and (**??**) match, which is good. Note that, here we are assuming A and $\vec{b}$ are real. Next, we can solve (**??**) to get $\widetilde{\vec{X}}$:

$$j\omega\widetilde{\vec{X}} = A\widetilde{\vec{X}} + \vec{b}\widetilde{U} \tag{10}$$

$$\Rightarrow (j\omega I - A)\widetilde{\vec{X}} = \vec{b}\widetilde{U} \tag{11}$$

$$\Rightarrow \widetilde{\vec{X}} = (j\omega I - A)^{-1}\vec{b}\widetilde{U} \tag{12}$$

3. Low-pass Filter (34 pts)

You have a 1 kΩ resistor and a 1 μF capacitor wired up as a low-pass filter.

(a) (4 pts) **Draw the filter, labeling the input node, output node, and ground.**

Solution:

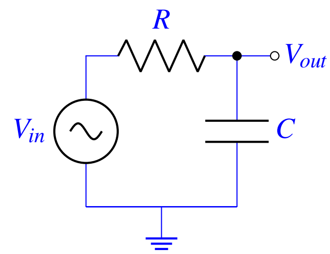


Figure 6: A simple RC circuit

(b) (6 pts) **Write down the transfer function of the filter, $H(\omega)$.** Be sure to use the given values for the components.

Solution: First, we convert everything into the phasor domain. We have,

$$Z_R = R = 1 \times 10^3 \Omega \tag{13}$$

$$Z_L = j\omega C = j\omega \times 10^{-6}\mathrm{F} \tag{14}$$

In phasor domain, we can treat these impedances essentially like we treat resistors. Hence,

$$\widetilde{V}_{out} = \frac{Z_C}{Z_C + Z_R}\widetilde{V}_{in} \tag{15}$$

$$\frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{16}$$

$$= \frac{1}{1 + j\omega RC} \tag{17}$$

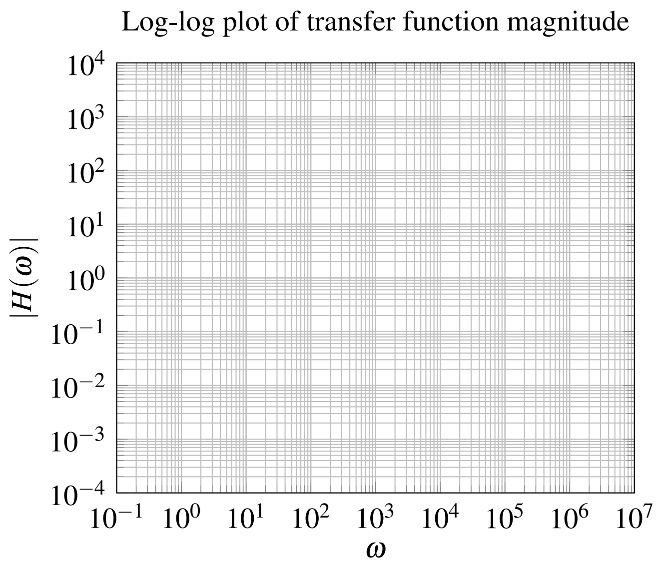
$$= \frac{1}{1 + j\omega / \frac{1}{RC}} \tag{18}$$

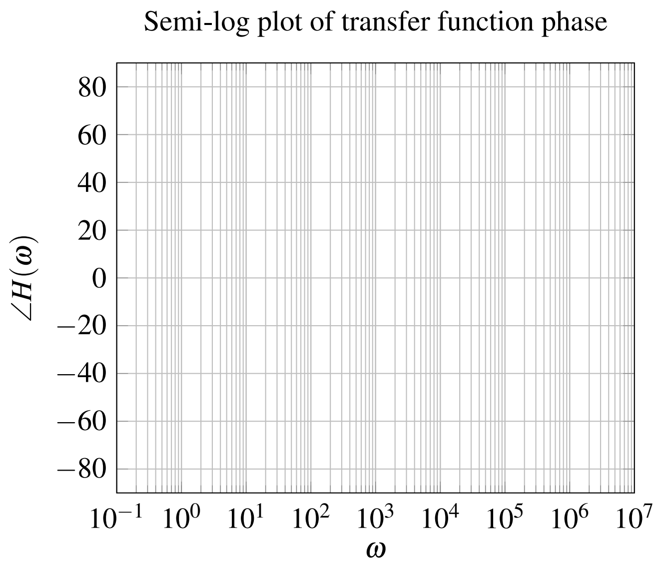
$$= \frac{1}{1 + j\omega \times 10^{-3}} \tag{19}$$

$$\tag{20}$$

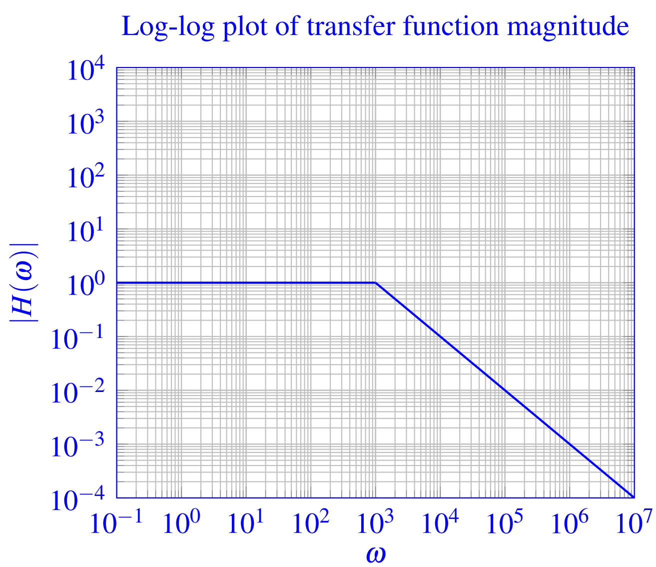
Hence, the corner frequency $\omega_C = \frac{1}{RC} = \frac{1}{10^3 \cdot 10^{-6}} = 10^3$ rad/sec.

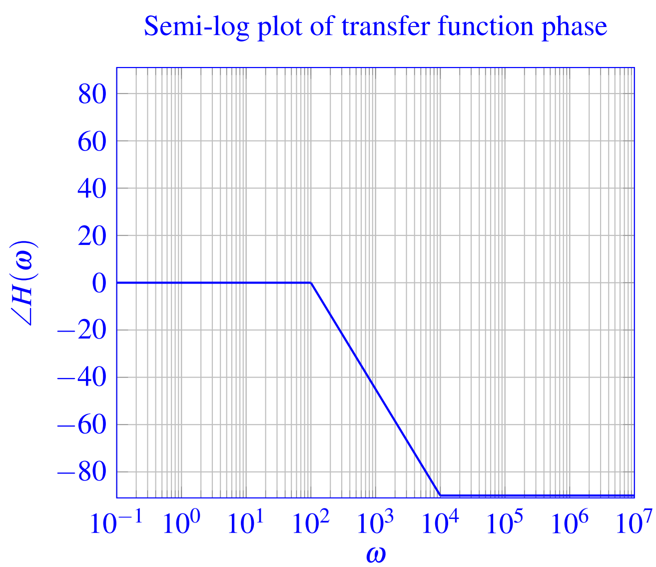
(c) (6 pts) **Draw a straight-line approximation to the Bode plot (both magnitude and phase) of the filter on the graph paper below.**





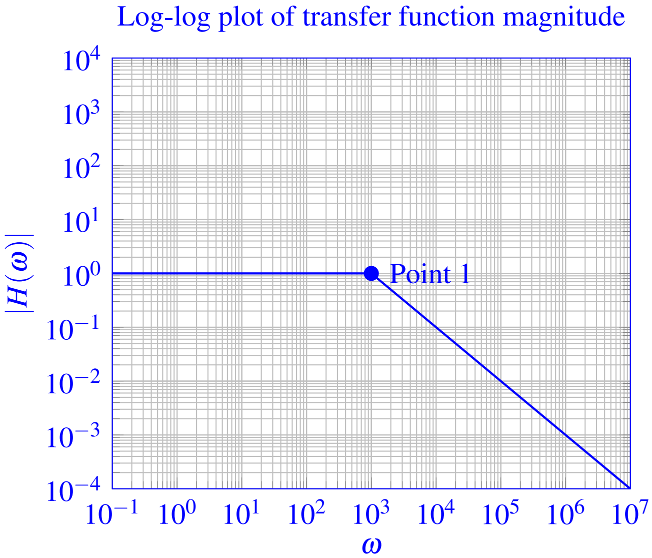
Solution:

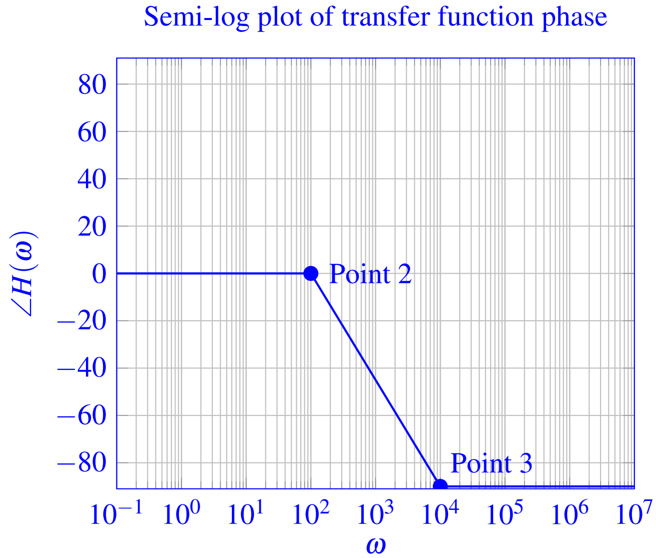




(d) (6 pts) **Annotate your Bode plot with three circles, each representing where the straight line approximation has its worst errors.** One circle should be on the magnitude plot, and two should be on the phase plot and corresponds to an absolute error. **Label each circle with the error at that point** (multiplicative error in the case of the magnitude plot, and absolute error in terms of the phase plot). For the phase plot, feel free to use trigonometric functions if you want.

Solution:





We can rewrite our transfer function to be in the form $H(\omega) = X(\omega) + jY(\omega)$:

$$H(\omega) = \frac{1}{1 + j\omega/\omega_c} \cdot \frac{1 - j\omega/\omega_c}{1 - j\omega/\omega_c} = \frac{1 - j\omega/\omega_c}{1 + \omega^2/\omega_c^2}$$

$$H(\omega) = \frac{1}{1 + \omega^2/\omega_c^2} - j\frac{\omega/\omega_c}{1 + \omega^2/\omega_c^2}$$

Now we can write the amplitude and phase of $H(\omega)$:

$$|H(\omega)| = \sqrt{\left(\frac{1}{1 + \omega^2/\omega_c^2}\right)^2 + \left(-\frac{\omega/\omega_c}{1 + \omega^2/\omega_c^2}\right)^2} \tag{21}$$

$$= \frac{\sqrt{1 + \omega^2/\omega_c^2}}{1 + \omega^2/\omega_c^2} \tag{22}$$

$$\angle H(\omega) = \text{atan2}(-\frac{\omega}{\omega_c}, 1)$$

Errors are:

- Point 1: Magnitude greatest error at $\omega = \omega_c = 10^3$ rad/sec
  The actual value of $|H(\omega = 10^3)| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$.
  The Bode plot approximation is $|H_{Bode}(\omega = 10^6)| \approx 1$.
  The absolute error at this point is:

  $$error_1 = \left|1 - \frac{1}{\sqrt{2}}\right|$$

  The multiplicative error e_M at this point is:

  $$e_M \cdot |H(\omega)|_{actual} = |H(\omega)|_{Bode}$$

  $$e_M = \frac{|H(\omega)|_{Bode}}{|H(\omega)|_{actual}} = \sqrt{2}$$

- Point 2: Phase greatest error at $\omega = 0.1 * \omega_c = 10^2$ rad/sec
  The actual value of $\angle H(\omega = 10^2) = \text{atan2}(-0.1, 1) \approx -6°$.
  The Bode plot approximation is $\angle H_{Bode}(\omega = 10^2) \approx 0°$.
  The absolute error at this point is:

$$error_2 = \left|0° - \text{atan2}(-0.1, 1)\right| = \left|0° - \tan^{-1}(-0.1)\right| \approx |0° + 6°| \approx 6°$$

- Point 3: Phase greatest error at $\omega = 10 * \omega_c = 10^4$ rad/sec
  The actual value of $\angle H(\omega = 10^4) = \text{atan2}(-10, 1) \approx -84°$.
  The Bode plot approximation is $\angle H_{Bode}(\omega = 10^4) \approx -90°$.
  The absolute error at this point is:

$$error_2 = \left|-90° - \text{atan2}(-10, 1)\right| = \left|-90° - \tan^{-1}(-10)\right| \approx |-90° + 84°| \approx 6°$$

(e) (3 pts) **Write an exact expression for the *magnitude* of $H(\omega = 10^6)$, and give an approximate numerical answer.**

Solution: We obtained this expression for the transfer function's magnitude above:

$$|H(\omega)| = \frac{\sqrt{1 + \omega^2/\omega_c^2}}{1 + \omega^2/\omega_c^2}$$

Plugging in for $\omega = 10^6$:

$$|H(\omega = 10^6)| = \frac{\sqrt{1 + (10^6)^2/(10^3)^2}}{1 + (10^6)^2/(10^3)^2}$$

$$|H(\omega = 10^6)| = \frac{\sqrt{1 + 10^6}}{1 + 10^6}$$

Approximately:

$$|H(\omega = 10^6)| \approx \frac{\sqrt{10^6}}{10^6} = \frac{10^3}{10^6} = 10^{-3}$$

$$|H(\omega = 10^6)| \approx 10^{-3}$$

(f) (3 pts) **Write an exact expression for the *phase* of $H(\omega = 1)$, and give an approximate numerical answer.**

Solution: We obtained this expression for the transfer function's phase above:

$$\angle H(\omega) = \text{atan2}(-\frac{\omega}{\omega_c}, 1)$$

Plugging in for $\omega = 1$:

$$\angle H(\omega = 1) = \text{atan2}(-\frac{10^0}{10^3}, 1) = \tan^{-1}(-10^{-3})$$

By the small angle approximation, this is:

$$\angle H(\omega = 1) \approx 10^{-3°}$$

(g) (6 pts) **Write down an expression for the time-domain output waveform $V_{out}(t)$ of this filter if the input voltage is $V(t) = 1\sin(1000t)$ V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

Solution: We can find the transfer function at this point:

$$|H(\omega = 10^3)| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\angle H(\omega = 10^3) = \text{atan2}(-1, 1) = -45^\circ$$

Therefore the output will be:

$$V_{out}(t) = \frac{1}{\sqrt{2}}\sin(1000t - 45^\circ)$$

4. What Is The Use of a Ferrite Bead? (32 pts)

You've probably noticed how most laptop chargers have a short, thick section near the end that plugs into the laptop.

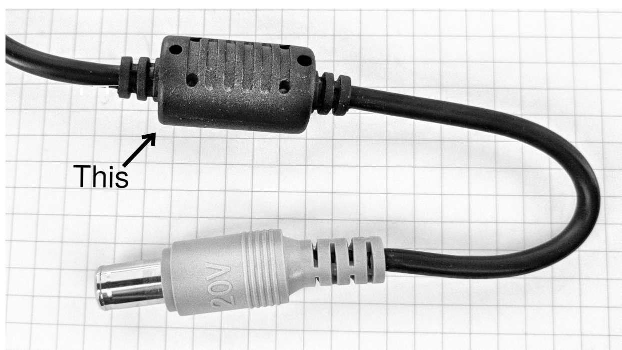


Figure 7: The short, thick section in question.

It's called a *ferrite bead*. It's a small shell of magnetic material called *ferrite*, and it makes the section of the wire that it surounds into **more** of an inductor (all wires have some inherent inductance, and in long wires, this is not negligible). Its purpose is to help filter out power supply noise. In circuit terms, the setup looks like this:

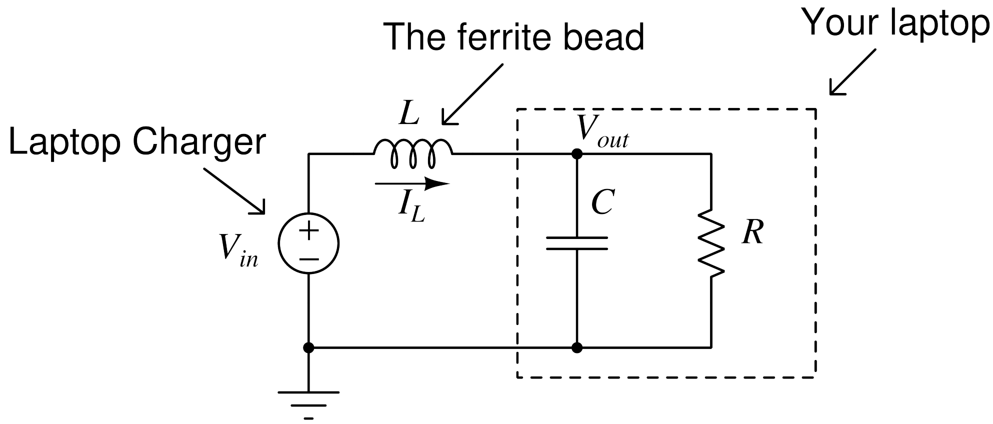


Figure 8: The ferrite bead in its natural habitat. L includes the wire inductance as well.

Here, V_{in} is the voltage produced by the charger, and V_{out} is the voltage that reaches the laptop. The resistor models the power consumption of the laptop, and the capacitor is part of an internal power supply filter.

In power supply filter design, the *time-domain* behavior of the filter is just as important as the *phasor-domain* behavior. However, because of exam-length limitations, we will just do the time-domain part here.

(a) (15 pts) Using $x_1(t) = I_L(t)$ and $x_2(t) = V_{out}(t)$ as state variables, **construct a matrix differential equation**

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}V_{in}(t) \tag{23}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. i.e. What are A and $\vec{b}$ for this circuit?

Solution:

To solve this part, we need to find two differential equations using circuit analysis, and then put them into matrix form.

To find one of the differential equations, we can use KVL around the loop comprising the voltage source, the ferrite bead, and the capacitor. Going clockwise, KVL gives

$$-V_{in}(t) + V_L(t) + V_C(t) = 0. \tag{24}$$

Applying $V_L(t) = L\frac{d}{dt}I_L(t)$ gives

$$-V_{in}(t) + L\frac{d}{dt}I_L(t) + V_C(t) = 0, \tag{25}$$

which we can write as

$$\frac{d}{dt}I_L(t) = -\frac{1}{L}V_C(t) + \frac{1}{L}V_{in}(t). \tag{26}$$

We now have one differential equation, so we just need to apply one more.

To find another differential equation, we can apply KCL at the $V_C(t)$ node. This gives

$$I_L(t) = I_C(t) + I_R(t). \tag{27}$$

Applying $I_C(t) = C\frac{d}{dt}V_C(t)$ and $I_R = \frac{1}{R}V_R(t) = \frac{1}{R}V_C(t)$ gives

$$I_L(t) = C\frac{d}{dt}V_C(t)\frac{1}{R}V_C(t), \tag{28}$$

which we can write as

$$\frac{d}{dt}V_C(t) = \frac{1}{C}I_L(t) - \frac{1}{RC}V_C(t). \tag{29}$$

Now we have a system of two differential equations, namely the system

$$\frac{d}{dt}I_L(t) = -\frac{1}{L}V_C(t) + \frac{1}{L}V_{in}(t) \tag{30}$$

$$\frac{d}{dt}V_C(t) = \frac{1}{C}I_L(t) - \frac{1}{RC}V_C(t). \tag{31}$$

Using the state variable names we assigned, we can write this as

$$\frac{d}{dt}x_1(t) = -\frac{1}{L}x_2(t) + \frac{1}{L}V_{in}(t) \tag{32}$$

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t) - \frac{1}{RC}x_2(t). \tag{33}$$

In matrix form, this is

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{\vec{b}} V_{in}(t). \tag{34}$$

We can read A and $\vec{b}$ from this equation, that is

$$A = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \tag{35}$$

$$\vec{b} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}. \tag{36}$$

(b) (8 pts) **What are the eigenvalues of the matrix A you found in the previous part?** Your answer should be in terms of R, L, and C.

Solution: To find the eigenvalues of the A matrix from the previous part, we need to find the solutions of $\det(A - \lambda I) = 0$; that is, the λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} - \lambda \end{bmatrix}\right) \tag{37}$$

$$= (-\lambda)\left(-\frac{1}{RC} - \lambda\right) - \left(-\frac{1}{L}\right)\left(\frac{1}{C}\right) \tag{38}$$

$$= \lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0. \tag{39}$$

The Quadratic formula gives

$$\lambda = -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}. \tag{40}$$

Now we have an expression for the eigenvalues λ in terms of R, L, and C, so you can stop here. Any further simplification or evaluation of the expression above is acceptable as well, of course.

(c) (6 pts) The reason that we care about the time-domain behavior of this circuit is that *we must prevent resonance*. Although LC resonance is often useful, in a power supply it must be avoided to prevent dangerous high-voltage oscillations from occurring.

How can we tell if the filter we design will resonate or not? Well, any imaginary component to the eigenvalues can induce oscillation. Using the eigenvalues you found in the previous part, **find the smallest value for L such that the system will not oscillate. Your answer should be in terms of R and C.**

Solution:

As the problem text suggests, to ensure that oscillation does not occur we need to ensure that both eigenvalues are purely real. This will be the case as long as the quantity in the square root in the expression for λ is nonnegative, that is as long as

$$\frac{1}{R^2C^2} - \frac{4}{LC} \geq 0. \tag{41}$$

To get an expression for L, we just have to isolate it, which we can do in the following steps:

$$\frac{1}{R^2C^2} - \frac{4}{LC} \geq 0 \tag{42}$$
$$\frac{4}{LC} \leq \frac{1}{R^2C^2} \tag{43}$$
$$\frac{1}{L} \leq \frac{1}{4R^2C} \tag{44}$$
$$L \geq 4R^2C. \tag{45}$$

Notice that, between Eq. (**??**) and Eq. (**??**), the direction of the inequality changes. This happened because we took the reciprocal of both sides of the inequality (think about it).

So now we know that, to prevent oscillation, we need to choose L such that $L \geq 4R^2C$. Of course, the *smallest* L satisfying this inequality is just the L when the equality holds. So, we can say that the smallest L that prevents oscillation is

$$L = 4R^2C. \tag{46}$$

(d) (3 pts) Suppose $R = 4\,\Omega$, $C = 10\,\mu\text{F}$. Using these values, **choose the smallest value for L that does not allow for LC resonance.**

Solution: For this part, we just need to take the formula $L = 4R^2C$ from the previous part and substitute in the given values for R and C. This gives

$$L = 4(4\,\Omega)^2(10\,\mu\text{F}) = (64 \times 10 \times 10^{-6})\text{H} = 640 \times 10^{-6}\text{H} = 640\,\mu\text{H}. \tag{47}$$

5. Transistor Switch Model (30 pts)

You have two CMOS inverters made from NMOS and PMOS devices. Both NMOS and PMOS devices have an "on resistance" of $R_{on} = 1$ kΩ, and each has a gate capacitance (input capacitance) of $C = 1$ fF (femto-Farads = 10^{-15}). We assume the "off resistance" (the resistance when the transistor is off) is infinite (*i.e.*, the transistor acts as an open circuit when off). The supply voltage V_{DD} is 1V. The two inverters are connected in series, with the output of the first inverter driving the input of the second inverter.

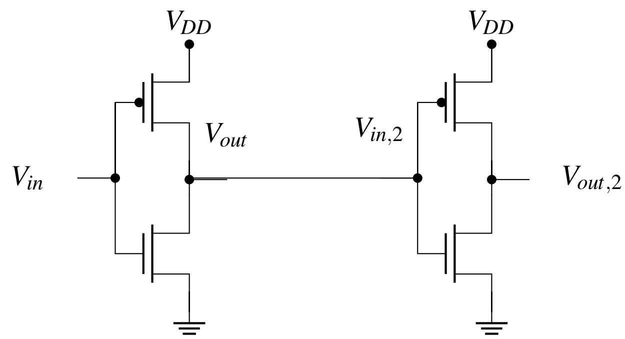


Figure 9: CMOS Inverter

(a) (14 pts) Assume the input to the first inverter has been low ($V_{in} = 0$ V) for a long time, and then switches at time $t = 0$ to high ($V_{in} = V_{DD}$). **Draw a simple RC circuit and write a differential equation describing the output voltage of the first inverter for time $t \geq 0$.** Don't forget that the second inverter is "loading" the output of the first inverter — you need to think about both of them.

(b) (6 pts) **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the initial slope, (3) the asymptotic value, and (4) the time that it takes for the voltage to decay to roughly 1/3 of its initial value.**

Solution:

(c) (6 pts) A long time later, the input to the first inverter switches low again. **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the initial slope, and (3) the asymptotic value.**

Solution:

(d) (4 pts) For each complete input cycle described in the two steps above, **how much charge is pulled out of the power supply?** Give both a symbolic and numerical answer.

Solution:

Contributors:

- Regina Eckert.
- Jaijeet Roychowdhury.
- Anant Sahai.
- Kris Pister.
- Alex Devonport.