EECS 16B Designing Information Devices and Systems II
Summer 2020 UC Berkeley

Signals Review

1. DFT Properties

(a) Show that the k^{th} frequency component of a length N signal $x[n]$ can be written as

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Solution: The DFT of a signal $x[n]$ can be written as $X[k] = Fx[n]$. If we look at the k^{th} entry of X, this is equivalent to taking the inner product of the k^{th} row of F and $\vec{x}$.

$$X[k] = \langle \vec{f}_k, \vec{x} \rangle \qquad \vec{f}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & e^{-j\frac{2\pi}{N}k} & e^{-j\frac{2\pi}{N}k\cdot 2} & \dots & e^{-j\frac{2\pi}{N}k\cdot(N-1)} \end{bmatrix}^T \tag{1}$$

Therefore, it follows that

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

(b) Given the DFT $X[k]$ of a time domain signal $x[n]$, show that

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

Solution: The inverse DFT of $X[k]$ can be written as $x[n] = F^* X[k] = UX[k]$. If we look at the n^{th} entry of x, this is equivalent to taking the inner product of the n^{th} row of U and $\vec{x}$.

$$x[n] = \langle \vec{u}_n, \vec{X} \rangle \qquad \vec{u}_n = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & e^{j\frac{2\pi}{N}n} & e^{j\frac{2\pi}{N}n\cdot 2} & \dots & e^{j\frac{2\pi}{N}n\cdot(N-1)} \end{bmatrix}^T \tag{2}$$

Therefore, it follows that

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

(c) Prove that if $x[n]$ is a real valued signal, $X[k] = \overline{X[N-k]}$.

Solution: We now know from part (a) that

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

This means that $X[-k] = X[N-k]$ will be

$$X[N-k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(N-k)\cdot n}$$
$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}k\cdot n}$$

Therefore the complex conjugate of $X[N-k]$ will be

$$\overline{X[N-k]} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = X[k]$$

(d) Prove that if $x[n]$ is real and $x[n] = x[N-n]$, then all of the DFT coefficients $X[k]$ are real.

Solution: Let start with the defintion of the DFT and look at the conjugate $\overline{X[k]}$.

$$\overline{X}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \overline{x[n]} e^{j\frac{2\pi}{N}kn}$$

Since $x[n]$ is real and $x[n] = x[N-n]$ we shall substitute $\overline{x[n]} = x[N-n]$.

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[N-n] e^{j\frac{2\pi}{N}kn}$$

Defining the variable $m = N-n$, we change variables in our summation to

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} x[m] e^{j\frac{2\pi}{N}k(N-m)}$$

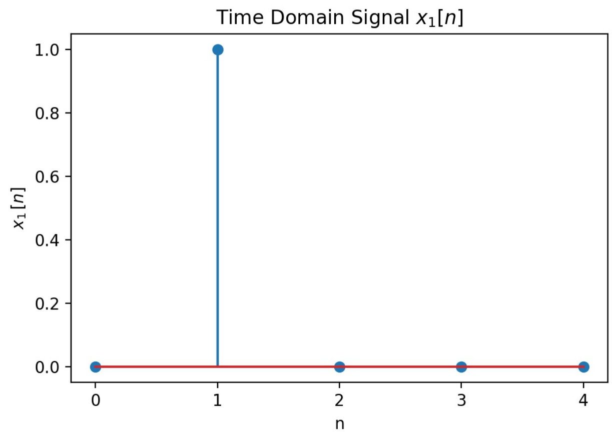
Since $e^{j2\pi} = 1$, it follows that

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}m} = X[k]$$

2. DFT Basics

Compute the 5 point DFT of the following signals

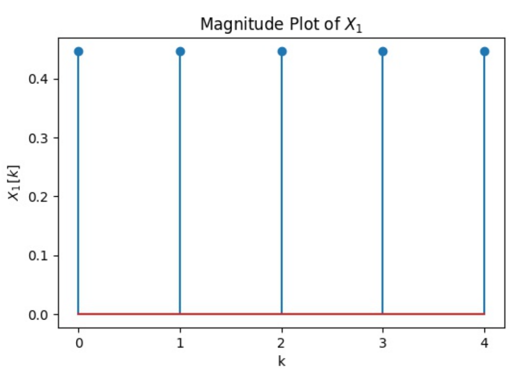
(a) $x_1[n] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

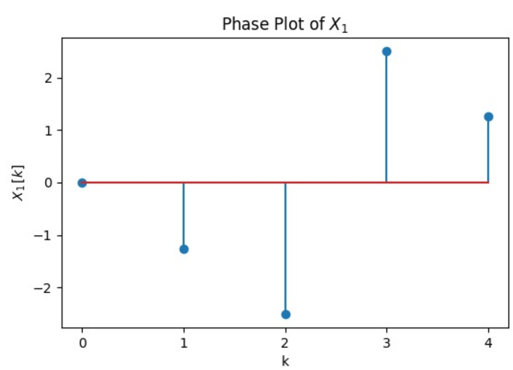


Solution: We can compute the frequency components by multiplying by the matrix $F = U^*$. Since $x_1[n]$ is zero for $\neq 1$, the frequency components will be the second column of F.

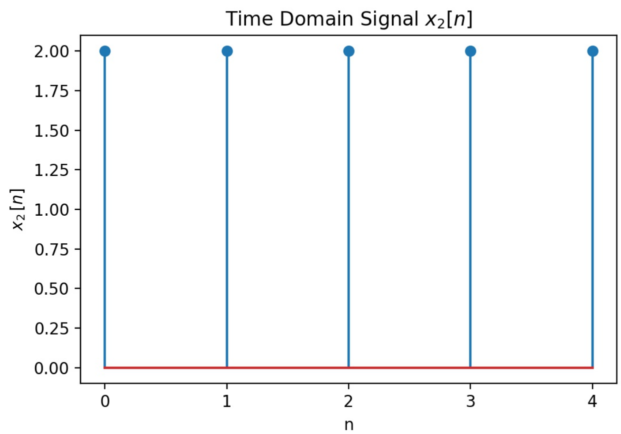
$$X_1[k] = Fx_1[n] = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & e^{-j\frac{2\pi}{5}} & e^{-j\frac{2\pi}{5}\cdot 2} & e^{-j\frac{2\pi}{5}\cdot 3} & e^{-j\frac{2\pi}{5}\cdot 4} \end{bmatrix}$$

The magnitude and phase of X_1 as plotted below





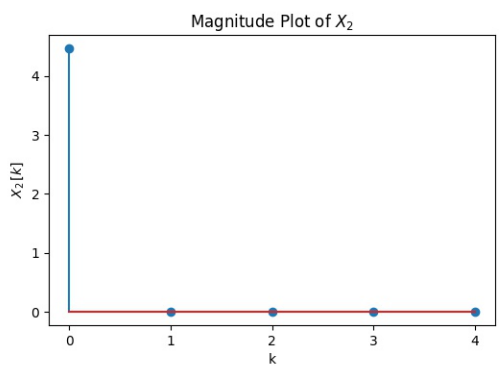
(b) $x_2[n] = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \end{bmatrix}$.

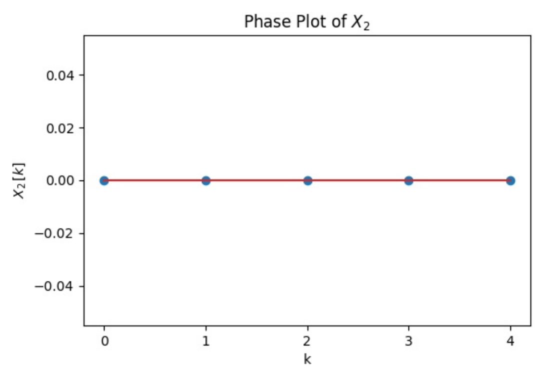


Solution: Since $x_2[n] = 2\sqrt{5}u_0[n]$ where u_0 is the DC component DFT basis vector, the frequency components must be

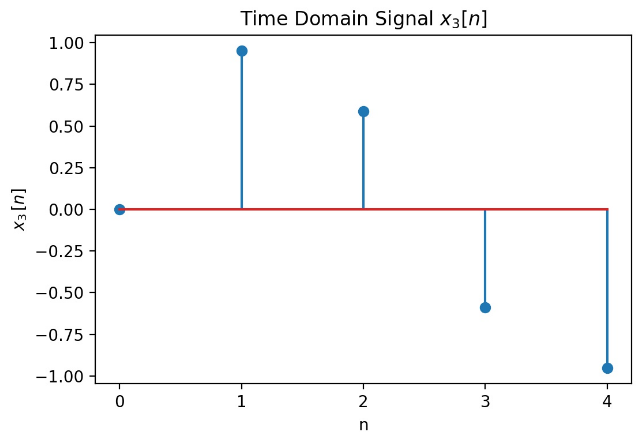
$$X_2[k] = \begin{cases} 2\sqrt{5} & k = 0. \\ 0 & k \neq 0. \end{cases}$$

The magnitude and phase of X_2 as plotted below





(c) $x_3[n] = \sin\left(\frac{2\pi}{5}n\right)$.

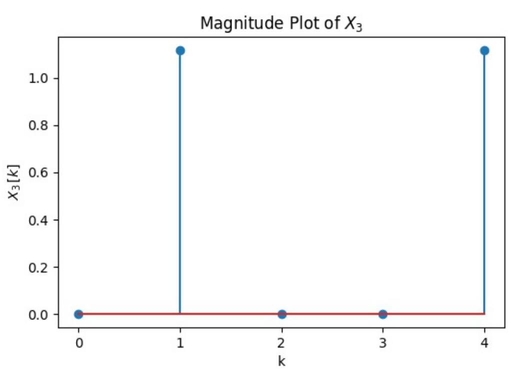


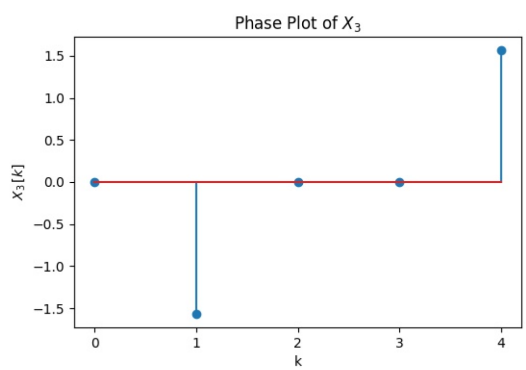
Solution:

$$\sin\left(\frac{2\pi}{5}n\right) = \frac{1}{2j}e^{j\frac{2\pi}{5}n} - \frac{1}{2j}e^{-j\frac{2\pi}{5}n}$$
$$u_k[n] = \frac{1}{\sqrt{5}}e^{j\frac{2\pi}{5}kn}$$
$$\vec{x}_3 = \frac{\sqrt{5}}{2j}(\vec{u}_1 - \vec{u}_4)$$

$$X_3[k] = \begin{cases} \frac{-\sqrt{5}j}{2} & k = 1 \\ \frac{\sqrt{5}j}{2} & k = 4 \\ 0 & k \neq 1, 4. \end{cases}$$

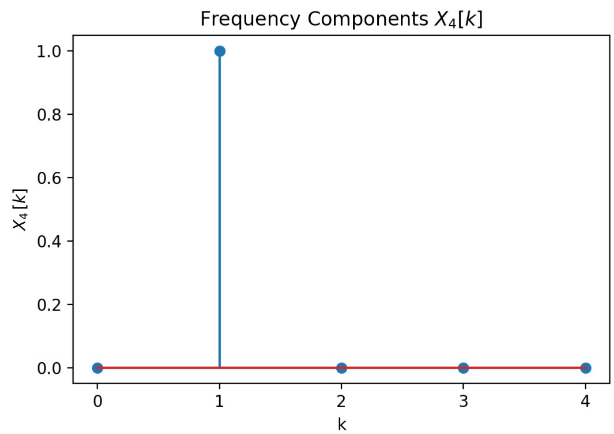
The magnitude and phase of X_3 as plotted below





Now compute the 5 point inverse DFT given the following frequency components

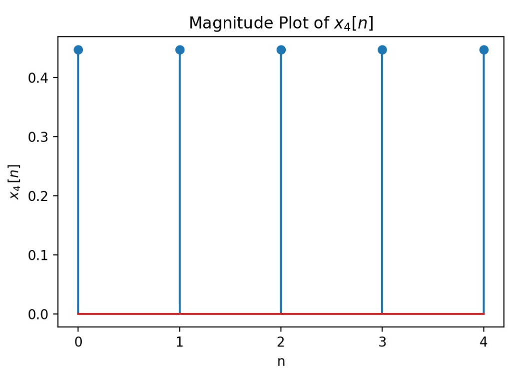
(d) $X_4[k] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

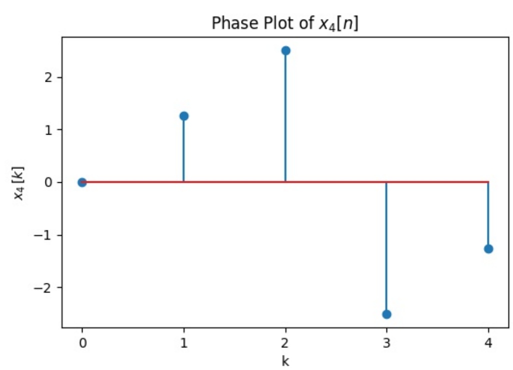


Solution: We can compute the time domain signal by multiplying by the matrix $F^* = U$. Since $x_4[n]$ is zero for $\neq 1$, the time components will be the second column of U.

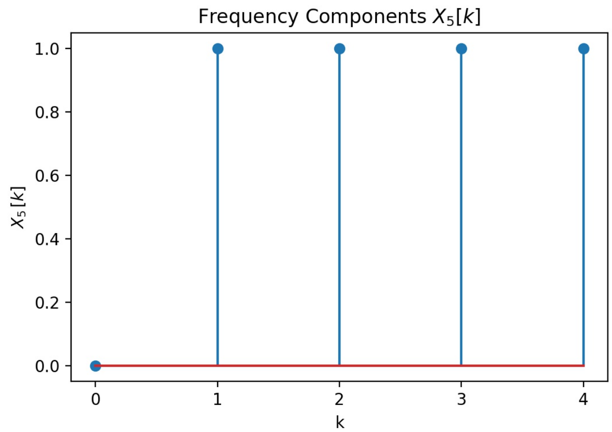
$$x_4[n] = UX_4[k] = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & e^{j\frac{2\pi}{5}} & e^{j\frac{2\pi}{5}\cdot 2} & e^{j\frac{2\pi}{5}\cdot 3} & e^{j\frac{2\pi}{5}\cdot 4} \end{bmatrix}$$

The magnitude and phase of $x_4[n]$ are plotted below





(e) $X_5[k] = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \end{bmatrix}$.



Solution: One way finding $x_5[n]$ is to realize that $X_5[n]$ is real and even meaning $x_5[n]$ is real and even. We'll show the derivation of its DFT in the next part.

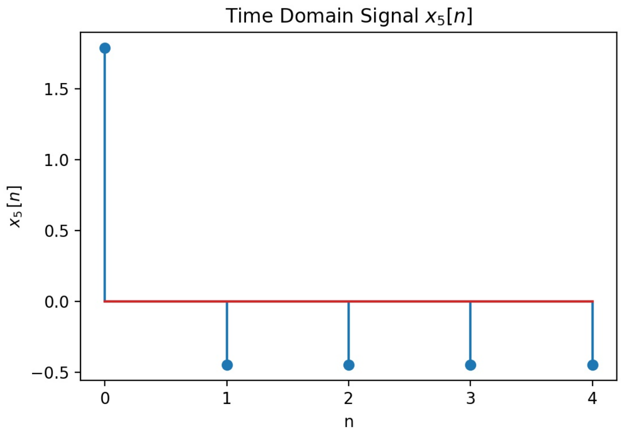
Alternatively, we can compute the IDFT using the summation formula

$$
\begin{aligned}
x[n] &= \frac{1}{\sqrt{5}} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{5}kn} \\
&= \frac{1}{\sqrt{5}} \left(e^{j\frac{2\pi}{5}n} + e^{j\frac{4\pi}{5}n} + e^{j\frac{6\pi}{5}n} + e^{j\frac{8\pi}{5}n} \right) = \frac{1}{\sqrt{5}} e^{j\frac{2\pi}{5}n} \frac{1 - e^{j\frac{8\pi}{5}n}}{1 - e^{j\frac{2\pi}{5}n}} && \text{Geometric Sum Formula} \\
&= \frac{1}{\sqrt{5}} e^{j\frac{2\pi}{5}n} \frac{e^{j\frac{4\pi}{5}n}}{e^{j\frac{\pi}{5}n}} \frac{e^{-j\frac{4\pi}{5}} - e^{j\frac{4\pi}{5}}}{e^{-j\frac{\pi}{5}n} - e^{j\frac{\pi}{5}n}} = \frac{1}{\sqrt{5}} e^{j\pi n} \frac{\sin\left(\frac{4\pi}{5}n\right)}{\sin\left(\frac{\pi}{5}n\right)} && \text{Factor to pull out a sin} \\
&= \frac{(-1)^n}{\sqrt{5}} \frac{\sin\left(\frac{4\pi}{5}n\right)}{\sin\left(\frac{\pi}{5}n\right)}
\end{aligned}
$$

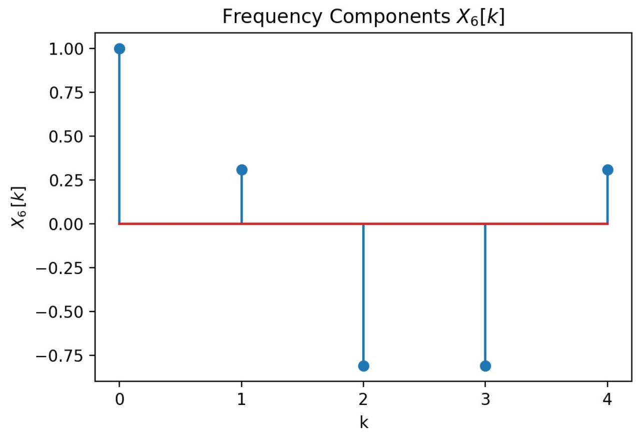
This is valid for all $n \neq 0$. When $n = 0$,

$$x[0] = \frac{1}{\sqrt{5}} \sum_{k=0}^{N-1} X[k] = \frac{4}{\sqrt{5}}$$

The real signal $x_5[n]$ is plotted below


Time Domain Signal $x_5[n]$
1.5
1.0
0.5
0.0
−0.5
$x_5[n]$
0
1
2
3
4
n

(f) $X_6[k] = \cos\left(\frac{2\pi}{5}k\right)$.



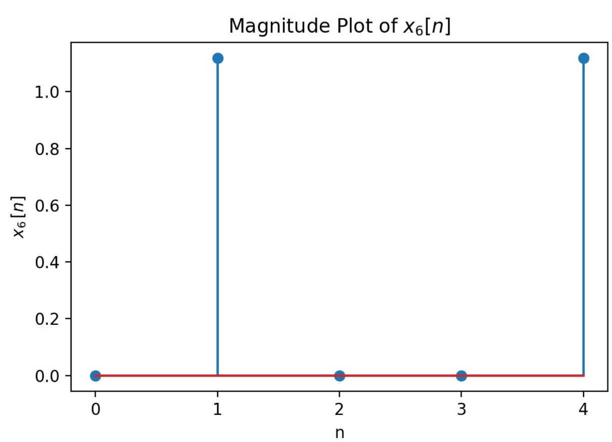
Solution: We can try to compute this using the IDFT matrix $F^* = U$. However, since X_6 is real and even, x_6 will be real and even. This implies that

$$
\begin{aligned}
x_6[n] &= F^* X_6[k] && \text{IDFT Formula} \\
x_6[n] &= \bar{F} X_6[k] && F \text{ is symmetric} \\
\overline{x_6[n]} &= F X_6[k] && \text{Conjugating both sides} \\
x_6[n] &= F X_6[k] && x[n] \text{ is real}
\end{aligned}
$$

Therefore, the IDFT of $X_6[k]$ is equal to its DFT.

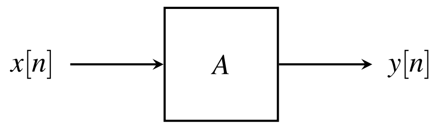
$$x_6[n] = \begin{cases} \frac{\sqrt{5}}{2} & k = 1,4 \\ 0 & k \neq 1,4. \end{cases}$$

The real signal $x_6[n]$ is plotted below



3. DFT and Finite Sequences (X points)

Consider a system $A\{\vec{x}\}$ which operates on length-8 sequences.



This system:

1) computes the DFT_8 of the sequence,
2) multiplies the first 4 elements ($k = 0, 1, 2, 3$) by $-j$ and the next 4 elements ($k = 4, 5, 6, 7$) by j, and
3) computes the IDFT_8 of the result.

(a) **Is the system linear?**

Solution: This system is **linear** since it can be modeled as matrix multiplications

$$\vec{y} = F^* \begin{bmatrix} -j & & & & & & & \\ & -j & & & & & & \\ & & -j & & & & & \\ & & & -j & & & & \\ & & & & j & & & \\ & & & & & j & & \\ & & & & & & j & \\ & & & & & & & j \end{bmatrix} F\vec{x}$$

We will refer to this diagonal matrix as D in the later parts.

(b) The system is applied on an input sequence $x[n] = \sin\left(\frac{\pi}{4}n\right)$, $0 \leq n < 8$. **What is $y[n]$, the output of the system?** Full credit will only be given to the simplest expression.

Solution: Since $x[n] = \frac{1}{2j}e^{j\frac{2\pi}{8}} - \frac{1}{2j}e^{-j\frac{2\pi}{8}}$, its DFT is

$$X[k] = \begin{bmatrix} 0 & \frac{\sqrt{8}}{2j} & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{8}}{2j} \end{bmatrix}$$

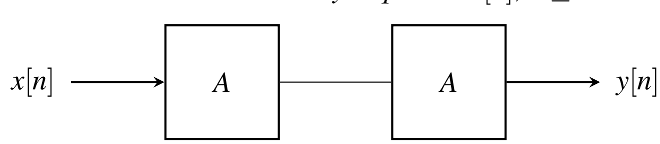
Applying the matrix multiplication, get

$$D[k] = \begin{bmatrix} 0 & -\frac{\sqrt{8}}{2} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{8}}{2} \end{bmatrix}$$

Lastly, taking the IDFT, we see that

$$x[n] = F^* D[k] = -\cos\left(\frac{\pi}{4}n\right)$$

(c) We apply two such systems in series to an *arbitrary sequence* $x[n]$, $0 \leq n < 8$:



Express $y[n]$ **in terms of** $x[n]$**.** Full credit will only be given to the simplest expression.

Solution:

$$\begin{aligned}\vec{y} &= A^2\vec{x} = F^*DFF^*DF\vec{x} \\ &= F^*D^2F\vec{x} = -F^*F\vec{x} \\ &= -\vec{x}\end{aligned}$$

Note that D^2 is a diagonal matrix of entries -1. Therefore we conclude that $y[n] = -x[n]$.

4. Integration by Convolution

Consider the following system that acts as a discrete-time integrator.

$$y[n]-y[n-1]=x[n] \tag{3}$$

We will assume that $y[n]=0$ for $n<0$.

(a) **Show that this system is LTI.**

Solution:

(i) **Linearity:**

- Scaling:
  Let $x[n]$ be an input with output $y[n]$. Then if we input $\hat{x}[n]=\alpha x[n]$,

  $$\hat{x}[n]=\alpha x[n]=\alpha(y[n]-y[n-1])=\alpha y[n]-\alpha y[n-1]$$

  This implies that $\hat{y}[n]=\alpha y[n]$.

- Additivity:
  Let $x_1[n]$ and $x_2[n]$ be inputs with outputs $y_1[n]$ and $y_2[n]$. Then if we input $\hat{x}[n]=(x_1+x_2)[n]$,

  $$\begin{aligned}\hat{x}[n]&=x_1[n]+x_2[n]=y_1[n]-y_1[n-1]+y_2[n]-y_2[n-1]\\&=y_1[n]+y_2[n]-y_1[n-1]-y_2[n-1]\end{aligned}$$

  This shows that $\hat{y}[n]=y_1[n]+y_2[n]$ is the output.

(ii) **Time-Invariance**

Let $\hat{x}[n]=x[n-n_0]$ be a delayed input signal. We see that

$$\hat{x}[n]=x[n-n_0]=y[n-n_0]-y[n-n_0-1]$$

As a result, the output $\hat{y}[n]$ must be $\hat{y}[n]=y[n-n_0]$.

(b) **What is the system's imuplse response?**

Solution:

$$\begin{aligned}h[0]-h[-1]&=\delta[0]\\h[n]-h[n-1]&=\delta[n]\quad\text{for } n>0\\\implies h[0]=1\qquad h[n]&=h[n-1]\quad\text{for } n>0\end{aligned}$$

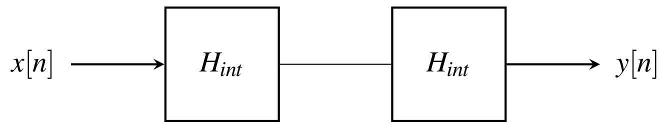
We conclude by saying that $h[n]$ is the unit step function $u[n]$.

(c) Suppose we input the unit step $x[n]=u[n]=\begin{cases}1 & n\geq 0\\0 & n<0\end{cases}$. **What is the output $y[n]$?**

Solution:

$$\begin{aligned}y[n]&=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=0}^{\infty}h[k]x[n-k] && \text{Convolution is commutative}\\&=\sum_{k=0}^{\infty}x[n-k] && h[k]=1\text{ for }k\geq 0.\\&=\sum_{k=0}^{n}x[k] && x[n-k]=0\text{ for }k>n.\\&=\sum_{k=0}^{n}1=n+1\end{aligned}$$

(d) Now let's create a new system of the following model



where each H_{int} represents one integrator system. **How can we express the input-output relationship of $x[n]$ and $y[n]$?**

Solution: The output of the first system is $y_1[n] = (x * h)[n]$. This is the output to the second system which will have output $y[n] = (y_1 * h_{int})[n] = ((x * h_{int}) * h_{int})[n]$.
Since convolution is associative, we can write out the input-output relation as

$$y[n] = x[n] * (h_{int} * h_{int})[n]$$

(e) **What is the impulse response of this new system?**

Solution: $\delta[n]$ is the convolution identity. Therefore, $h[n] = (\delta * (h_{int} * h_{int}))[n] = (h_{int} * h_{int})[n]$. We know that $h_{int}[n] = u[n]$ so from part (c), $h[n] = n+1$.

(f) If we input $x[n] = u[n]$ to this new system, **what would the output** $y[n]$ **be?**
Hint: What is the integrator system doing? If you aren't sure, look back at part (c).

Solution: The integrator system sums all of the values of $x[n]$ for $0 \leq k \leq n$. Therefore, the output to this system can be represented as

$$\begin{aligned} y[n] &= \sum_{i=1}^{n}(n+1) = \sum_{i=1}^{n} n + \sum_{i=1}^{n} 1 \\ &= \frac{(n+1)(n)}{2} + n + 1 = \frac{(n+2)(n+1)}{2} \end{aligned}$$

5. Stability of State Space Systems (X points)

Consider a discrete time state space system

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n].$$

For which of the following possible matrices $\mathbf{A}$ is the system stable? Explain your answers.

(a) **(X pts)**

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stable? Yes / No
Explanation:

Solution: $\lambda = 0, \frac{1}{2} \implies$ system is stable.

(b) **(X pts)**

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Stable? Yes / No
Explanation:

Solution: This is a circulant matrix of the signal $x[n] = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{bmatrix}$.
Therefore, its eigenvalues will the $\sqrt{N}$ times the DFT coefficients $X[k]$.

$$x[n] = \frac{1}{2}\cos\left(\frac{2\pi}{4}n\right) \implies X[k] = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad \lambda_4 = 1$$

Since $|\lambda_2| = 1$, the system is unstable.

For parts (c) and (d), consider a continuous time system

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = \mathbf{A}\vec{x}(t).$$

(c) **(X pts)**

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Stable? Yes / No
Explanation:

Solution: The matrix A has rank 1 meaning it has eigenvalues of 0. Therefore, the system is unstable.

(d) **(X pts)** Recall that we are still considering the continuous time system.

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & -1 \\ -1 & -2 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix}$$

Stable? Yes / No
Explanation:

Solution: This is a circulant matrix of the signal $x[n] = \begin{bmatrix} -2 & -1 & 0 & 1 \end{bmatrix}$.
Therefore, its eigenvalues will the $\sqrt{N}$ times the DFT coefficients $X[k]$.

$$X[0] = \frac{1}{2}\sum_{n=0}^{3} x[n] = -1$$

$$X[1] = \frac{1}{2}\sum_{n=0}^{3} x[n]e^{-j\frac{\pi}{2}n} = -1-j$$

$$X[2] = \frac{1}{2}\sum_{n=0}^{3} (-1)^n x[n] = -1$$

$$X[3] = \overline{X[1]} = -1+j$$

$$\lambda_1 = -2 \quad \lambda_2 = -2-2j \quad \lambda_3 = -2 \quad \lambda_4 = -2+2j$$

Since all eigenvalues have real part less than 0, the system is stable.

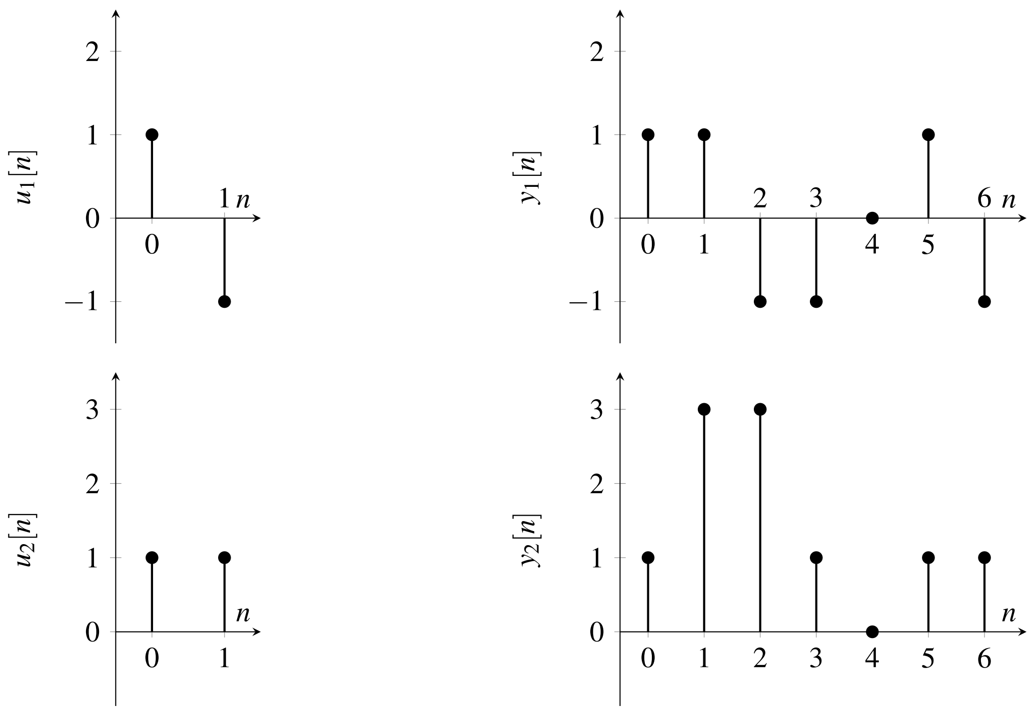
6. Signals and Systems (X points)

Consider a discrete time **observable** system

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \mathbf{B}u[n]$$
$$y[n] = \mathbf{C}\vec{x}[n],$$

where $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{B} \in \mathbb{R}^{N \times 1}$, and $\mathbf{C} \in \mathbb{R}^{1 \times N}$ are *unknown*. The system is in the state $\vec{x} = \vec{0}$ before any input is applied and is therefore LTI.

(a) **(X pts)** Given the following input-output pairs $u[n]$ and $y[n]$, what is the impulse response $h[n]$ of the system? Assume that the signals are 0 everywhere else.



Solution: Since the system is LTI and $\delta[n] = \frac{1}{2}(u_1[n] + u_2[n]), h[n] = \frac{1}{2}(y_1[n] + y_2[n])$.

$$h[n] = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(b) **(X pts) Is the system BIBO stable?**

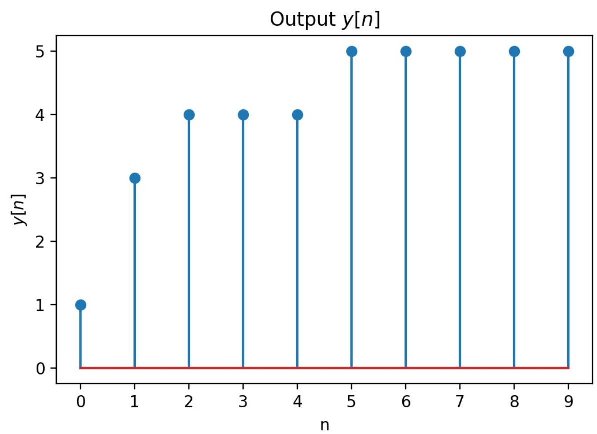
Solution: An LTI System is BIBO Stable iff

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

In our case, the sum is equal to 5 so the system must be stable.

(c) **(X pts)** Given the unit step input $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$, the system's output eventually reaches a steady state value. **At what time step does the output reach the steady state value and what is the steady state value of the output?**

Solution: If we compute the convolution $(u * h)[n]$, we get the following result



The steady state is $y = 5$ and the output reaches it at time $n = 5$.

7. Sampling and Interpolation

Consider a frequency source that produces a signal

$$x(t) = \cos(2\pi f_0 t).$$

This signal is sampled with a sampling interval of T_s [sec] and reconstructed as $\tilde{x}(t)$ using sinc interpolation.

(a) **What are the sampling intervals that will result in a constant $\tilde{x}(t)$ for all t?**

Solution: Aliasing of a pure frequency onto DC occurs when $f_s = \frac{f_0}{n}$ for $n = 1, 2, \ldots$. Therfore, $T_s = \frac{n}{f_0}$ for $n = 1, 2, \ldots$ and

$$x[n] = \cos(2\pi f_0 n T_s) = \cos(2\pi n) = 1$$

which will be interpolated to a constant.

(b) **How quickly must we sample $x(t)$ in order to get a perfect reconstruction?**

Solution: $\omega_{max} = 2\pi f_0$. From the Sampling Theorem, we know that if $\omega_{max} < \frac{\pi}{T_s}$, then the reconstruction will be perfect. This is equivalent to saying $T_s < \frac{1}{2f_0}$.

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