EECS 16B Designing Information Devices and Systems II
Summer 2020 UC Berkeley

SVD Review

1. Pick a Matrix

Given an example of a square matrix that satisfies the following conditions or prove that no such example can exist.

(a) (i) Can be diagonalized and is invertible.
(ii) Cannot be diagonalized but is invertible.
(iii) Can be diagonalized but is non-invertible.
(iv) Cannot be diagonalized and is non-invertible.

Solution: All of these matrices exist!

(i)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

(ii)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

(iii)

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(iv)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) (i) Has orthogonal columns and is invertible.
(ii) Has orthogonal columns but is non-invertible.
(iii) Has orthonormal columns and is diagonalizble.

Solution:

(i)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(ii) Remember that the zero vector is orthogonal to every vector.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

(iii)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Orthonormal T/F

(a) T If U is a matrix with orthonormal columns, then $U^*U = I$.

Solution: True, since $\vec{u}_i^*\vec{u}_j = 1$ if $i = j$ and $= 0$ for $i \neq j$.

(b) F If U is a matrix with orthornomal columns, then $UU^* = I$.

Solution: If U is not square then UU^* cannot have rank n so $UU^* \neq I$.

(c) T If U is matrix with orthonormal columns then $\|U\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{C}^n$.

Solution: True, since $U^*U = I$.

(d) F A matrix U with orthonormal columns has real eigenvalues.

Solution: False, take $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(e) T The singular values of a unitary matrix are all equal to 1.

Solution: True, since $U^*U = I$.

(f) F The eigenvalues of a unitary matrix are all equal to 1.

Solution: False, take the example from part (d)

3. Spectral T/F

(a) T The matrix A^*A is Hermitian.

Solution: True, since $(A^*A)^* = (A^*)(A^*)^* = A^*A$.

(b) F A symmetric matrix can have complex eigenvalues.

Solution: False, from the Spectral Theorem.

(c) F The matrix A^*A has positive eigenvalues.

Solution: False, A^*A may have an eigenvalue of zero. Only if A is full rank does A^*A have positive eigenvalues.

(d) T For a Hermitian matrix, the eigenvectors of distinct eigenvalue are orthogonal.

Solution: True, from the Spectral Theorem.

(e) F Linearly independent eigenvectors of the same eigenvalue of a Hermitian matrix are orthogonal.

Solution: False, take $A = I$. Then any nonzero vector $\vec{x}$ is an eigenvector.

(f) F The U and V matrices of the SVD of a Hermitian matrix are identical.

Solution: False, if the matrix has negative eigenvalues, then the vectors in the U and V matrix will have opposite signs.

4. SVD Stuff (X pts)

(a) Compute the SVD of the following matrix. Express your answer in the the form of $\sum_i \sigma_i \vec{u}_i \vec{v}_i^\top$

$$A = \begin{bmatrix} \vec{a} & -\vec{a} \end{bmatrix}$$

Here, $\vec{a}$ is some arbitrary vector in $\mathbb{R}^n$

Solution:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top$$

where $\vec{u}_1 = \frac{\vec{a}}{\|a\|}$, $\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, and $\sigma_1 = \|a\| * \sqrt{2}$

(b) Compute the compact form SVD of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Solution:

$$AA^\top = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

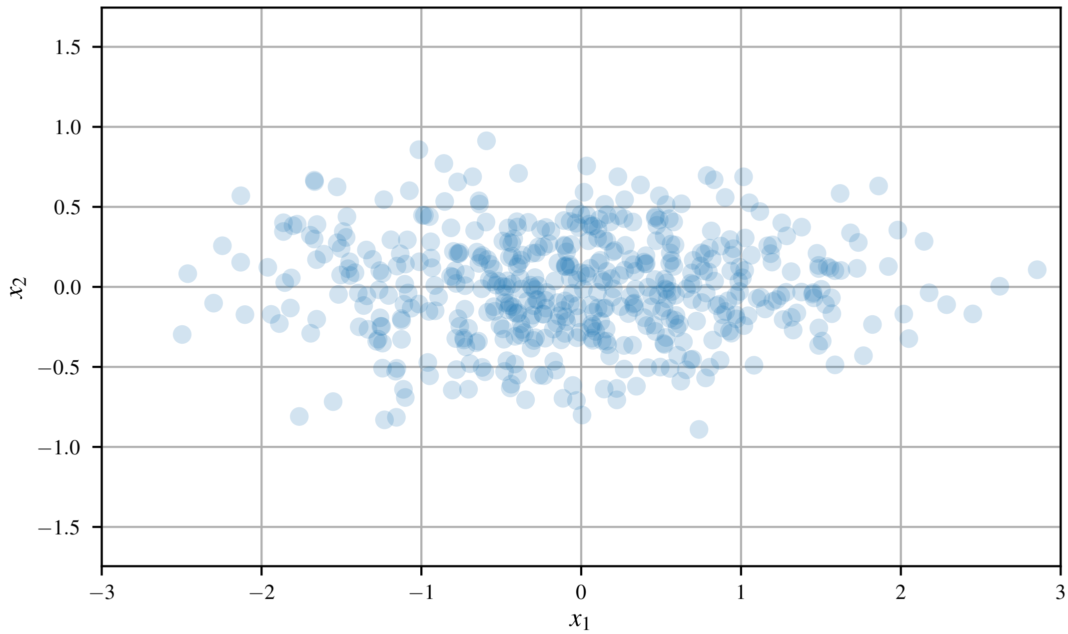
So the eigenvalues/eigenvectors of $AA^\top$ are $\lambda_1 = 9$, $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_2 = 1$, $\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

This means that $\sigma_1 = 3, \sigma_2 = 1$. To calculate $\vec{v}_i$, we can apply the formula $\vec{v}_i = \frac{1}{\sigma_i} A^\top \vec{u}_i$. Thus,

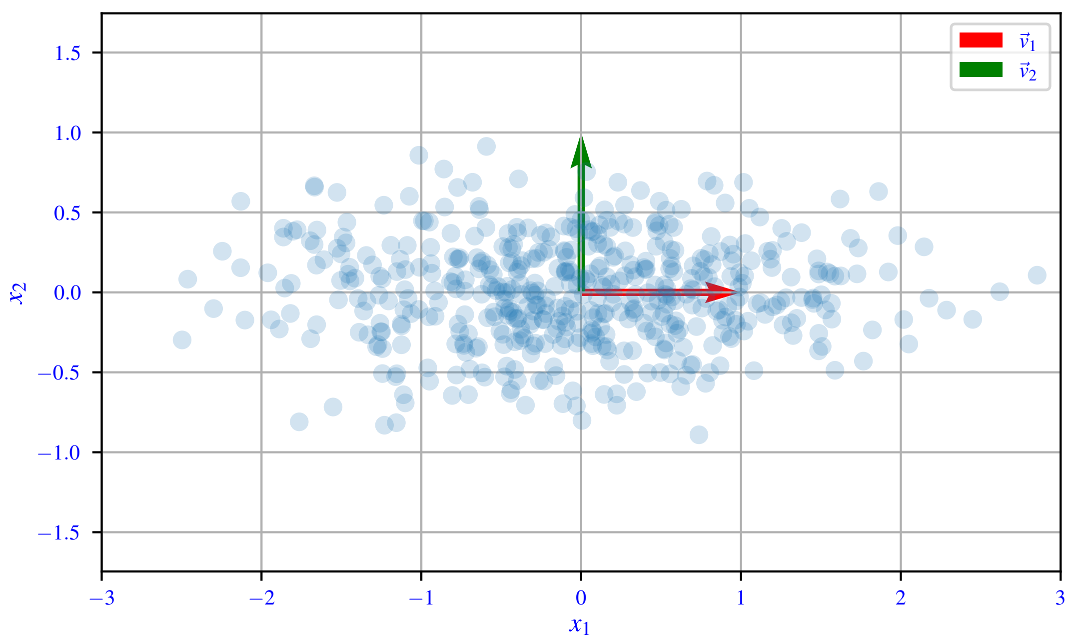
$$\vec{v}_1^\top = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 1 \end{bmatrix}$$

$$\vec{v}_2^\top = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

(c) Consider an A matrix where each row is a vector in $\mathbb{R}^2$ that corresponds to one point in the plot below:



On the plot above, draw your best estimate for $\vec{v}_1$ and $\vec{v}_2$.



Solution:

5. Spectral Norm Proof

How can we measure the size of a matrix? One way to think about this is to look at the ratio $\frac{\|A\vec{x}\|}{\|\vec{x}\|}$ over all vectors $\vec{x}$. In fact, the **Spectral Norm** of a matrix $\|A\|_2$ can be defined as

$$\|A\|_2 = \max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$$

In this problem we will try to find what the value of $\|A\|_2$ is and show that it in fact is related to the SVD. Let's start by noting that $A^T A$ is symmetric and has eigenvalue decomposition $V \Lambda V^T$.

(a) **For $\vec{x} \in \mathbb{R}^n$, decompose $\vec{x}$ as a linear combination of the set of orthonormal eigenvectors of $A^T A$.**

Solution: In order to write $\vec{x}$ as a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_n$, we must find scalars α_i where

$$\vec{x} = \alpha_1 \vec{v}_1 + \ldots \alpha_n \vec{v}_n$$

If we take the inner product of $\vec{x}$ with $\vec{v}_i$, since the eigenvectors are orthonormal, we will get:

$$\langle \vec{x}, \vec{v}_i \rangle = \alpha_i \langle \vec{v}_i, \vec{v}_i \rangle = \alpha_i$$

Therefore we can write out $\vec{x}$ as:

$$\begin{aligned}\vec{x} &= \langle \vec{x}, \vec{v}_1 \rangle \vec{v}_1 + \ldots \langle \vec{x}, \vec{v}_n \rangle \vec{v}_n \\ &= \sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i\end{aligned}$$

(b) **Express $\|A\vec{x}\|^2$ in terms of $\vec{v}_i$, $\vec{x}$, and σ_i,** for $i \in \{1, 2, \ldots, n\}$.

Solution:

$$\begin{aligned}\|A\vec{x}\|^2 &= (A\vec{x})^T A\vec{x} = \vec{x}^T A^T A\vec{x} \\ &= \left(\sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i^T\right) A^T A \left(\sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i\right) \\ &= \left(\sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i^T\right)\left(\sum_{i=1}^{n} \lambda_i \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i\right) \\ &= \sum_{i=1}^{n} \lambda_i \langle \vec{x}, \vec{v}_i \rangle^2\end{aligned}$$

We know that $\lambda_i = \sigma_i^2$ so we finish up by expressing $\|A\vec{x}\|^2$ as

$$\|A\vec{x}\|^2 = \sum_{i=1}^{n} \sigma_i^2 \langle \vec{x}, \vec{v}_i \rangle^2$$

(c) **Find a unit vector $\vec{x}$ that maximizes $\|A\vec{x}\|^2$.**

Solution: Since $\vec{x}$ is a unit vector, we know $\|\vec{x}\| = 1$. Using our expression for $\vec{x}$ in part (a), we can write out its norm as:

$$
\begin{aligned}
\|\vec{x}\|^2 &= \vec{x}^T\vec{x} \\
&= \left(\sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i\rangle \vec{v}_i\right)^T \left(\sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i\rangle \vec{v}_i\right) \\
&= \sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i\rangle^2 = 1
\end{aligned}
$$

If we let $\alpha_i = \langle \vec{x}, \vec{v}_i\rangle^2$, we see that $\alpha_i > 0$,

$$
\sum_{i=1}^{n} \alpha_i = 1 \text{ and } \|A\vec{x}\|^2 = \sum_{i=1}^{n} \alpha_i \sigma_i^2
$$

meaning $\|A\vec{x}\|^2$ is a linear combination of σ_i^2 with α_i as the scalars summing up to 1.
Since $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, the $\|A\vec{x}\|^2$ is maximized when $\alpha_1 = 1$ and all the other $\alpha_i = 0$. Note that $\alpha_1 = 1$ is equivalent to saying that $\langle \vec{x}, \vec{v}_1\rangle^2 = 1$ so $\vec{x} = \pm\vec{v}_1$.

(d) **Show that $\|A\vec{x}\| \leq \sigma_1\|x\|$ for any $x \in \mathbb{R}^n$. Thus, $\|A\|_2 = \sigma_1$.**

Solution: From previous parts, we know

$$
\begin{aligned}
\|A\vec{x}\|^2 &= \sum_{i=1}^{n} \sigma_i^2 \langle x, v_i\rangle^2 \\
&\leq \sigma_1^2 \sum_{i=1}^{n} \langle x, v_i\rangle^2 \\
&= \sigma_1^2 \|x\|^2.
\end{aligned}
$$

Thus,

$$
\|A\vec{x}\| \leq \sigma_1\|x\|.
$$

And when $\vec{x} = \pm\|x\|\vec{v}_1$, the maximum of $\frac{\|A\vec{x}\|}{\|x\|}$ is reached:

$$
\max \frac{\|A\vec{x}\|}{\|x\|} = \sigma_1.
$$

(e) **Show that $\|A\vec{x}\| \geq \sigma_n\|x\|$ for any $x \in \mathbb{R}^n$. Thus, $\min \frac{\|A\vec{x}\|}{\|x\|} = \sigma_n$.**

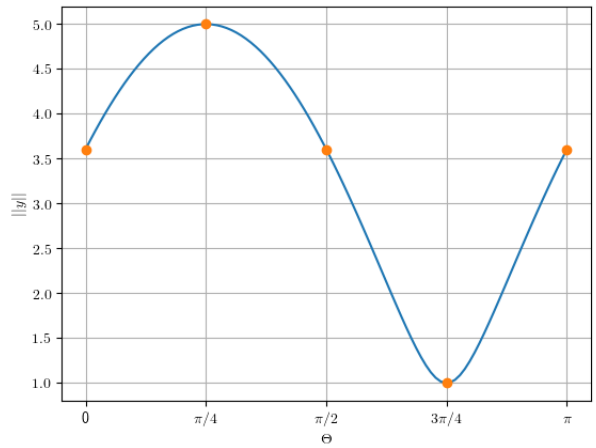
Solution: We can set up the same problem we did in part (b) and realize that

$$
\begin{aligned}
\|A\vec{x}\|^2 &= \sum_{i=1}^{n} \sigma_i^2 \langle x, v_i\rangle^2 \\
&\geq \sigma_n^2 \sum_{i=1}^{n} \langle x, v_i\rangle^2 \\
&= \sigma_n^2 \|x\|^2.
\end{aligned}
$$

The norm $\|A\vec{x}\|$ is minimized when $\vec{x} = \pm\|x\|\vec{v}_n$. Thus $\min \frac{\|A\vec{x}\|}{\|x\|} = \sigma_n$.

6. SVD (X points)

(a) Let $A \in \mathbb{R}^{2\times 2}$ and $\vec{x} = \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$, $||\vec{x}|| = 1$. Now let $\vec{y} = A\vec{x}$. Below is the plot of $||\vec{y}||$ vs θ.



A has the SVD $U\Sigma V^\top$. Either specify what the matrices U, Σ, and V are; or state they they cannot be determined from the information given.

Solution: We know that $\sigma_2 \le ||A\vec{x}|| \le \sigma_1$, so from the above graph we can see that $\sigma_1 = 5$ and $\sigma_2 = 1$.

These occur for $\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \vec{v}_1$ and $\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \vec{v}_2$ respectively. Since we observe $||\vec{y}||$, $\vec{y}$ can be arbitrarily rotated by U, so we cannot deduce a unique U.

(b) Let $A \in \mathbb{R}^{N\times N}, B \in \mathbb{R}^{N\times N}$ be full rank matrices and let $\vec{x} \in \mathbb{R}^N$ have $||\vec{x}|| = 1$. Let $\vec{y} = AB\vec{x}$.
Find a upper bound for $||\vec{y}||$ in terms of the singular values of A and B. Explain your answer.
Solution:

$$||B\vec{x}|| \le \sigma_{max}\{A\}$$
$$\left\| A \frac{\vec{x}}{||\vec{x}||} \right\| \le \sigma_{max}\{A\}$$

Where $\sigma_{max}\{M\}$, for some matrix M, is the largest singular value of M.
If $\vec{x} = \vec{v}_1\{B\}$ and $B\vec{x} = \vec{v}_1\{A\}$, then the output is maximal, with

$$||AB\vec{x}|| = \sigma_{max}\{A\} \cdot \sigma_{max}\{B\}$$

7. Low Rank Approximation

Given a $m \times n$ matrix A, of high rank, we want to see how we can best approximate this matrix A using a lower rank matrix A_k of rank $k << n$.

To measure this Low-Rank approximation, we will look at the following norm: $\|A - B_k\|_2$ where B_k is a matrix of rank k. In this problem, we will are interested in the following optimization problem

$$\min_{B_k} \|A - B_k\|_2$$
$$\text{subject to } \text{Rank}(B_k) \leq k$$

and show that the optimal B_k is in fact A_k or the rank k SVD approximation of A:

$$A_k = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^T \tag{1}$$

(a) **What is the spectral norm of** $\|A - A_k\|_2$?

Solution: We know from the SVD that $A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i \vec{u}_i \vec{v}_i^T$. Therefore, $A - A_k = \sum_{i=k+1}^{n} \sigma_i \vec{u}_i \vec{v}_i^T$. As a result, the spectral norm of $A - A_k$ will be its largest singular value, which in this case will be σ_{k+1}.

To show A_k is optimal, we must show that $\|A - B_k\|_2 \geq \|A - A_k\|_2 = \sigma_{k+1}$ for any matrix B_k of rank k.

To do this, we will first consider a vector $\vec{y} = \alpha_1 \vec{v}_1 + \cdots + \alpha_{k+1} \vec{v}_{k+1}$ that is a linear combination of the first $k+1$ vectors of the V matrix of the SVD of A. We will also define a subspace $S = \text{span}\{\vec{v}_1, \ldots, \vec{v}_{k+1}\}$ and show that there must exist a vector $\vec{y}$ in both S and $\text{Nul}(B_k)$.

(b) **What are the dimensions of the Nul**(B_k) **and** S?

Solution: By the Rank-Nullity Theorem, we know that $\text{Rank}(B_k) + \dim \text{Nul}(B_k) = n$. Since $\text{Rank}(B_k) = k$, we see that $\dim \text{Nul}(B_k) = n - k$.

Since $S = \text{span}\{\vec{v}_1, \ldots, \vec{v}_{k+1}\}$, and all of the vectors $\vec{v}_i$ are linearly independent, $R = \{\vec{v}_1, \ldots, \vec{v}_{k+1}\}$ forms a basis for S meaning S has dimension $k+1$.

(c) **Show that there must exist a** $\vec{y} = \alpha_1 \vec{v}_1 + \ldots + \alpha_{k+1} \vec{v}_{k+1}$ **that is in both subspaces Nul**(B_k) **and** S.
Hint: Let R be a basis for S. Then create a basis B for the Nul(B_k) and look at the union of the two bases.

Solution: We know that $R = \{\vec{v}_1, \ldots, \vec{v}_{k+1}\}$ is a basis for S. Since $\text{Nul}(B_k)$ has dimension $n-k$, we can pick a basis $B = \{\vec{w}_1, \ldots, \vec{w}_{n-k}\}$ for $\text{Nul}(B_k)$. Now if we look at $R \cup B$, this is a set of $n+1$ vectors that are in $\mathbb{R}^n$, so this set must be linearly dependent. This means that we can write $\vec{w}_{n-k}$ as a linear combination of the remaining vectors:

$$\vec{w}_{n-k} = \alpha_1 \vec{v}_1 + \ldots + \alpha_{k+1} \vec{v}_{k+1} + \beta_1 \vec{w}_1 + \ldots + \beta_{n-k-1} \vec{w}_{n-k-1}.$$

Subtracting over the $\vec{w}_i$ vectors, we see that

$$\alpha_1 \vec{v}_1 + \ldots + \alpha_{k+1} \vec{v}_{k+1} = -\beta_1 \vec{w}_1 - \ldots - \beta_{n-k-1} \vec{w}_{n-k-1} + \vec{w}_{n-k}.$$

As a result, $\vec{y} = \alpha_1 \vec{v}_1 + \ldots + \alpha_{k+1} \vec{v}_{k+1}$ is a linear combination of the vectors in R so it must be in S. It is also however, a linear combination of the vectors in B so it must also be in $\text{Nul}(B_k)$.

(d) Let $\|\vec{y}\| = 1$ and **show that** $\|A - B_k\|_2 \geq \left\|(A - B_k)\vec{y}\right\|$.

Solution: Using the definition of the spectral norm, we see that

$$\|A - B_k\|_2 = \max_{\vec{x} \neq 0} \frac{\left\|(A - B_k)\vec{x}\right\|}{\|\vec{x}\|}$$

If we put the constraint that $\|\vec{x}\| = 1$, then

$$\|A - B_k\|_2 = \max_{\|\vec{x}\|=1} \left\|(A - B_k)\vec{x}\right\|.$$

which implies that

$$\|A - B_k\|_2 \geq \left\|(A - B_k)\vec{y}\right\|$$

(e) **Express** $\|\vec{y}\|^2$ **and** $\|A\vec{v}_i\|^2$ **in terms of** $\alpha_1, \ldots, \alpha_{k+1}$.

Solution: Using the definition of a norm with respect to the inner product we get:

$$\|\vec{y}\|^2 = \langle \vec{y}, \vec{y} \rangle = \langle \alpha_1 \vec{v}_1 + \ldots + \alpha_{k+1}\vec{v}_{k+1}, \alpha_1 \vec{v}_1 + \ldots + \alpha_{k+1}\vec{v}_{k+1} \rangle$$

Then we can use the distributive properties of norms, and the fact that $\vec{v}_i$ are orthonormal to cancel the cross-terms.

$$\|\vec{y}\|^2 = \sum_{i=1}^{k+1} \langle \alpha_i \vec{v}_i, \alpha_i \vec{v}_i \rangle = \sum_{i=1}^{k+1} \alpha_i^2 \langle \vec{v}_i, \vec{v}_i \rangle = \sum_{i=1}^{k+1} \alpha_i^2$$

For the $\|A\vec{v}_i\|$, we again apply the definition of a norm with respect to the inner product to get:

$$\begin{aligned}
\|A\vec{v}_i\|^2 &= \langle A\vec{v}_i, A\vec{v}_i \rangle = (A\vec{v}_i)^T (A\vec{v}_i) \\
&= \vec{v}_i^T A^T A \vec{v}_i = \vec{v}_i^T (\lambda_i \vec{v}_i) = \lambda_i \vec{v}_i^T \vec{v}_i \\
&= \lambda_i = \sigma_i^2
\end{aligned}$$

Therefore $\|A\vec{v}_i\| = \sigma_i$.

(f) **Simplify** $\left\|(A - B_k)\vec{y}\right\|$ **and conclude that** $\|A - B_k\|_2 \geq \sigma_{k+1}$.

Solution: Since $\vec{y}$ is in $\text{Nul}(B_k)$, we can simplify $\left\|(A - B_k)\vec{y}\right\|$ as:

$$\left\|(A - B_k)\vec{y}\right\| = \|A\vec{y} - B_k\vec{y}\| = \|A\vec{y}\|$$

Plugging in for $\vec{y} = \alpha_1 \vec{v}_1 + \ldots + \alpha_{k+1}\vec{v}_{k+1}$ and looking at $\|A\vec{y}\|^2$, we get:

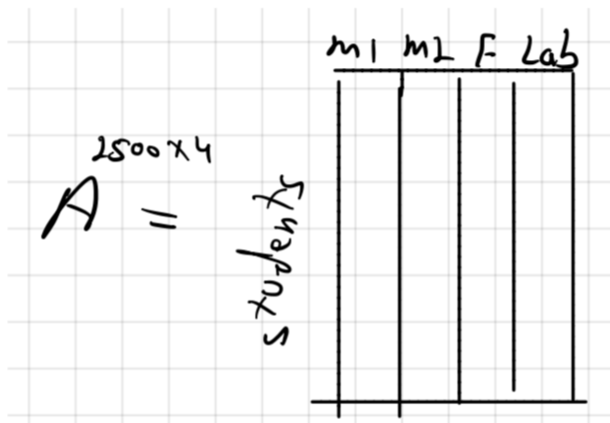
$$\begin{aligned}
\|A\vec{y}\|^2 &= \langle \alpha_1 A\vec{v}_1 + \ldots + \alpha_{k+1} A\vec{v}_{k+1}, \alpha_1 A\vec{v}_1 + \ldots + \alpha_{k+1} A\vec{v}_{k+1} \rangle \\
&= \alpha_1^2 \langle A\vec{v}_1, A\vec{v}_1 \rangle + \ldots + \alpha_{k+1}^2 \langle A\vec{v}_{k+1}, A\vec{v}_{k+1} \rangle \\
&= \alpha_1^2 \sigma_1^2 + \ldots + \alpha_{k+1}^2 \sigma_{k+1}^2
\end{aligned}$$

However, since $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{k+1}$ and $\alpha_1^2 + \ldots + \alpha_{k+1}^2 = 1$, $\|A\vec{y}\|^2 \geq \sigma_{k+1}^2$. Therefore, we conclude by saying that $\|A\vec{y}\| \geq \sigma_{k+1}$ which implies $\|A - B_k\|_2 \geq \sigma_{k+1}$ proving the optimality of A_k.

8. PCA Midterm question

This question comes from fa17 midterm 2.

Consider a matrix $A \in \mathbb{R}^{2500\times 4}$ which represents the EE16B Sp'2020 midterm 1, midterm 2, final and lab grades for all 2500 students taking the class.

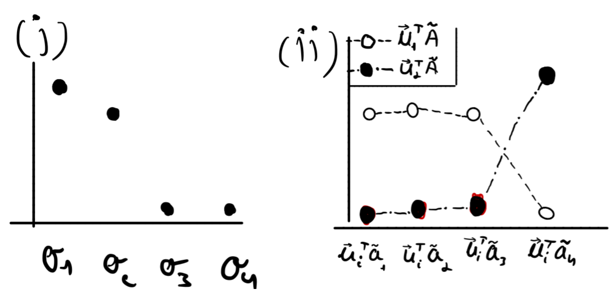


To perform PCA, you subtract the mean of each column and store the results in $\tilde{A}$. Your analysis includes:

(a) Computing the SVD: $\tilde{A} = \sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T + \sigma_3\vec{u}_3\vec{v}_3^T + \sigma_4\vec{u}_4\vec{v}_4^T$ and plot the singular values.

(b) Computing the graph $\vec{u}_1^T\tilde{A}$ and $\vec{u}_2^T\tilde{A}$

The analysis data are plotted below:



Based on the analysis, answer the following true or false questions. Briefly explain your answer.

(a) __T__ The data can be approximated well by two principle components.

Solution:
True. From graph (i), there are two significant singular values, the rest are small.

(b) __T__ The students' exam scores have significant correlation between the exams.

Solution: True. $\vec{u}_i^T\tilde{a}_i$ gives the correlation between the ith principle component and the variable stored in the ith column of $\tilde{A}$. From graph (ii), both midterms as well as final grades are highly correlated with the first principle component while all three also have little to no correlation to the second principle component. This means the exam scores are highly correlated with each other.

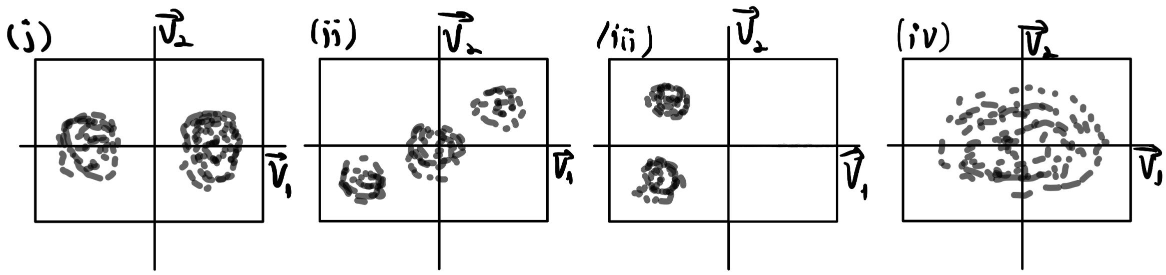
(c) _F_ The middle plot (ii) shows that students who did well on the exam did not do well in the labs and vice versa.

Solution: False. Graph (ii) shows that the exam scores and lab scores are not correlated at all. i.e. we cannot determine a student's lab score from their exam scores and vice versa.

(d) _T_ One of the principle components attributes is solely associated with lab scores and not with exam scores.

Solution: True. From graph (ii), the second principle component is only highly correlated with the lab scores.

(e) Circle all the scatter plots that could describe the data projected on the largest two principle components.



Solution: Plots (i) and (iv) are possible projections. Plot (ii) has more variance along the diagonal axis of $\vec{v}_1$ and $\vec{v}_2$ which contradicts the choice of $\vec{v}_1$ as the direction of maximal variance. Plot (iii) is not centered around the origin.

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