Discussion 2B

The following sections from the textbook are useful for this discussion: Sections 4.1, 4.3, 4.4

1. RL Transient Analysis (Hambley Example 4.2)

Consider the circuit shown in Figure 1.

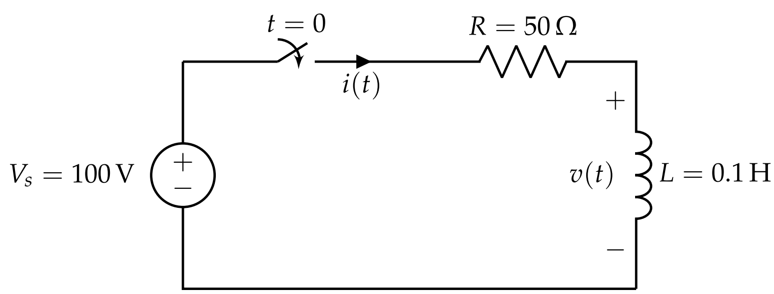

Figure 1

(a) **Set up a differential equation for the current $i(t)$ in the form:**

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + ai(t) = b(t) \tag{1}$$

and determine the initial condition $i(0)$.

Solution: Before the switch closes ($t < 0$), we have that the switch is open and thus current is zero.

$$i(t) = 0 \quad \text{for } t < 0 \tag{2}$$

Furthermore, since inductor current cannot change instantaneous/must be continuous, we know that immediately after the switch is closed, $i(0) = 0$. We will use this initial condition when solving our differential equation later on.

To begin solving for current $i(t)$, we will write out the KVL equation for the loop, which states that the sum of the voltages around a loop is 0:

$$V_s - V_R - V_L = 0 \tag{3}$$

$$V_R + V_L = V_s \tag{4}$$

$$Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = V_s \tag{5}$$

Rearranging our differential equation to be in the desired form, we get:

$$Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = V_s \tag{6}$$

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) = V_s \tag{7}$$

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{R}{L}i(t) = \frac{V_s}{L} \tag{8}$$

where $a = \frac{R}{L}$ and $b(t) = \frac{V_s}{L}$.

(b) **Solve for the current $i(t)$ using the integrating factor method.**

Solution: Approach 1: Integrating Factor Method

Starting from the equation we derived above:

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{R}{L}i(t) = \frac{V_s}{L} \tag{9}$$

where $a = \frac{R}{L}$ and $b(t) = \frac{V_s}{L}$. From here, we know that the integrating factor $f(t)$ is

$$\begin{aligned} f(t) &= \mathrm{e}^{\int a \mathrm{d}t} && (10) \\ &= \mathrm{e}^{\int \frac{R}{L} \mathrm{d}t} && (11) \\ &= \mathrm{e}^{\frac{R}{L}t} && (12) \end{aligned}$$

Multiplying both sides of our differential equation by this factor, we get:

$$\begin{aligned} \mathrm{e}^{\frac{R}{L}t}\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \mathrm{e}^{\frac{R}{L}t}\frac{R}{L}i(t) &= \mathrm{e}^{\frac{R}{L}t}\frac{V_s}{L} && (13) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(i(t)\mathrm{e}^{\frac{R}{L}t}\right) &= \frac{V_s}{L}\mathrm{e}^{\frac{R}{L}t} && (14) \\ i(t)\mathrm{e}^{\frac{R}{L}t} &= \int \frac{V_s}{L}\mathrm{e}^{\frac{R}{L}t'}\,\mathrm{d}t' && (15) \\ i(t)\mathrm{e}^{\frac{R}{L}t} &= \frac{V_s}{L}\int \mathrm{e}^{\frac{R}{L}t'}\,\mathrm{d}t' && (16) \\ i(t)\mathrm{e}^{\frac{R}{L}t} &= \frac{V_s}{L}\left(\frac{L}{R}\mathrm{e}^{\frac{R}{L}t}\right) + c && (17) \\ i(t)\mathrm{e}^{\frac{R}{L}t} &= \frac{V_s}{R}\left(\mathrm{e}^{\frac{R}{L}t}\right) + c && (18) \\ i(t) &= \frac{V_s}{R} + c\mathrm{e}^{-\frac{R}{L}t} && (19) \end{aligned}$$

Now, we will use our initial conditions to solve for the constant c.

$$\begin{aligned} i(0) &= \frac{V_s}{R} + c\mathrm{e}^{-\frac{R}{L}0} && (20) \\ 0 &= \frac{V_s}{R} + c && (21) \\ c &= -\frac{V_s}{R} && (22) \end{aligned}$$

Putting this all together, we get that our final equation for current is:

$$\begin{aligned} i(t) &= \frac{V_s}{R} - \frac{V_s}{R}\mathrm{e}^{-\frac{R}{L}t} && (23) \\ i(t) &= 2 - 2e^{-500t} && (24) \end{aligned}$$

Approach 2: Pattern-Matching

Recall that the general form of the solution of a differential euqation in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + ay(t) = b(t) \tag{25}$$

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is

$$y(t) = A\mathrm{e}^{-at} + \mathrm{e}^{-at} \int \mathrm{e}^{at} b(t)\,\mathrm{d}t \tag{26}$$

Plugging this into the general form and solving gives us:

$$i(t) = Ae^{-\frac{R}{L}t} + e^{-\frac{R}{L}t} \int e^{\frac{R}{L}t} \frac{V_s}{L}\,\mathrm{d}t \tag{27}$$

$$i(t) = Ae^{-\frac{R}{L}t} + \frac{V_s}{L} e^{-\frac{R}{L}t} \int e^{\frac{R}{L}t}\,\mathrm{d}t \tag{28}$$

$$i(t) = Ae^{-\frac{R}{L}t} + \frac{V_s}{L} e^{-\frac{R}{L}t} \left(\frac{L}{R} e^{-\frac{R}{L}}\right) t \tag{29}$$

$$i(t) = Ae^{-\frac{R}{L}t} + \frac{V_s}{R} \tag{30}$$

Lastly, we will use the initial condition $i(t) = 0$ to solve for A:

$$i(0) = Ae^{-\frac{R}{L}0} + \frac{V_s}{R} \tag{31}$$

$$0 = A + \frac{V_s}{R} \tag{32}$$

$$A = -\frac{V_s}{R} \tag{33}$$

Therefore, our final equation is also:

$$i(t) = \frac{V_s}{R} - \frac{V_s}{R}\mathrm{e}^{-\frac{R}{L}t} \tag{34}$$

$$i(t) = 2 - 2e^{-500t} \tag{35}$$

Approach 3: Textbook, "Ad hoc" Method

This form of a differential leads us to want to try a solution in the form of:

$$i(t) = K_1 + K_2\mathrm{e}^{st} \tag{36}$$

where we need to determine the values of the constants K_1, K_2, and s. We will first proceed by plugging in our trial solution into the differential equation which results in:

$$R\left(K_1 + K_2\mathrm{e}^{st}\right) + L\frac{\mathrm{d}}{\mathrm{d}t}\left(K_1 + K_2\mathrm{e}^{st}\right) = V_s \tag{37}$$

$$RK_1 + RK_2\mathrm{e}^{st} + L\left(sK_2\mathrm{e}^{st}\right) = V_s \tag{38}$$

$$RK_1 + (RK_2 + sLK_2)\mathrm{e}^{st} = V_s \tag{39}$$

By pattern matching the two sides of the equation: we have that:

$$K_1 = \frac{V_s}{R} = \frac{100}{50} = 2 \tag{40}$$

$$s = -\frac{R}{L} \tag{41}$$

If we plug this into our trial solution for current, we have:

$$i(t) = 2 + K_2\mathrm{e}^{-\frac{R}{L}t} \tag{42}$$

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Lastly, we can solve for the value of K_2 using the initial conditions. Using the initial condition that $i(0) = 0$ in, we get:

$$i(0) = 2 + K_2 e^{-\frac{R}{L}0} \tag{43}$$
$$0 = 2 + K_2 \tag{44}$$
$$K_2 = -2 \tag{45}$$

Hence, putting this all together, we have that the final solution for current is:

$$i(t) = 2 - 2e^{-\frac{R}{L}t} \quad \text{for } t > 0 \tag{46}$$

or if we replace $\tau = \frac{L}{R}$ where τ is the time constant, we get:

$$i(t) = 2 - 2e^{-\frac{t}{\tau}} \quad \text{for } t > 0 \tag{47}$$

(c) **Find the voltage $v(t)$.**

Solution: There are two approaches we can use to solve for $v(t)$. The first one is using the I-V relationsip of an inductor and the second is using the fact that $v(t)$ is also the difference between the source voltage and the resistor voltage.

Approach 1: We can use the equation:

$$v_L(t) = L\frac{di_L(t)}{dt} \tag{48}$$

which requires us to take the derivative of our derived current equation.

$$v(t) = L\frac{d}{dt}\left(-\frac{V_s}{R}e^{-\frac{R}{L}t}\right) \tag{49}$$
$$v(t) = L\frac{V_s}{R}\frac{R}{L}e^{-\frac{R}{L}t} \tag{50}$$
$$v(t) = L\frac{V_s}{R}\frac{R}{L}e^{-\frac{R}{L}t} \tag{51}$$
$$v(t) = V_s e^{-\frac{R}{L}t} \tag{52}$$
$$v(t) = 100e^{-500t} \tag{53}$$

Approach 2: The second approach involves recognizing that by KVL:

$$v(t) = V_s - V_R(t) \tag{54}$$

Leveraging Ohm's Law which states $V_R(t) = Ri(t)$, we can solve for $v(t)$:

$$v(t) = V_s - R\left(\frac{V_s}{R} - \frac{V_s}{R}e^{-\frac{R}{L}t}\right) \tag{55}$$
$$v(t) = V_s - V_s + V_s e^{-\frac{R}{L}t} \tag{56}$$
$$v(t) = V_s e^{-\frac{R}{L}t} \tag{57}$$
$$v(t) = 100e^{-500t} \tag{58}$$

which is the same solution as we got from Approach 1.

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2. Analyzing an RC Circuit with a Sinusoidal Source (Hambley Example 4.4)

Assume you are given the following circuit, where the capacitor is initially charged so that $v_C(t) = 1V$.

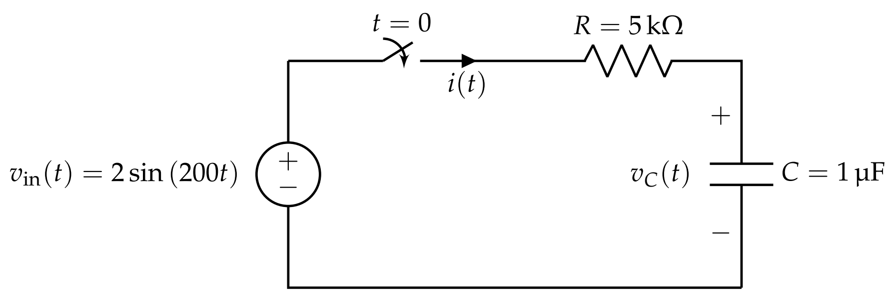

Figure 2

(a) **Set up a differential equation for the current $i(t)$ through the circuit in the form:**

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + ai(t) = b(t) \tag{59}$$

Solution: Let's start by writing the voltage equation for $t > 0$ using KVL:

$$v_R(t) + v_C(t) - v_{\text{in}}(t) = 0 \tag{60}$$

$$Ri(t) + \frac{1}{C}\int_0^t i(t')\,\mathrm{d}t' + v_C(0) - 2\sin(200t) = 0 \tag{61}$$

We wish to convert this to a differential equation, so let's take the derivative of the whole equation and then rearrange so that it is in the general form for a first-order differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Ri(t) + \frac{1}{C}\int_0^t i(t')\,\mathrm{d}t' + v_C(0) - 2\sin(200t)\right) = 0 \tag{62}$$

$$R\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}i(t) + 0 - 400\cos(200t) = 0 \tag{63}$$

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{RC}i(t) = \frac{400}{R}\cos(200t) \tag{64}$$

where $a = \frac{1}{RC} = \frac{1}{5 \times 10^{-3}} = 200$ and $b(t) = \frac{400}{R}\cos(200t) = 80 \times 10^{-3}\cos(200t)$.

(b) **Determine the initial condition of $i(t)$. In order words, solve for $i(0)$.**

Solution: Initially we have that $v_C(0) = 1\,\text{V}$. We know that at $t = 0$, $V_{\text{in}}(0) = 2\sin(200 \times 0) = 0\,\text{V}$. We can solve for the current using Ohm's Law:

$$i(0) = \frac{V_R(0)}{R} \tag{65}$$

$$i(0) = \frac{v_{\text{in}}(0) - v_C(t)}{R} \tag{66}$$

$$i(0) = \frac{0 - 1}{5 \times 10^3} \tag{67}$$

$$i(0) = -2 \times 10^{-4}\,\text{A} \tag{68}$$

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(c) **Solve for the current $i(t)$ through the circuit. Also, identify the transient response and the forced response of $i(t)$.** You may directly use the fact that the solution to a differential equation in the same form as Equation 59 is:

$$i(t) = Ae^{-at} + e^{-at}\int e^{at'} b(t')\,dt' \tag{69}$$

(HINT: The following integral might be useful:

$$\int e^{at}\cos(bt) = \frac{1}{b^2+a^2}e^{at}(b\sin(bt) + a\cos(bt)) \tag{70}$$

)

Solution: Now that we have our differential equation in the standard form, we can use the general form of the solution to a first-order differential equation to solve for $i(t)$. Recall that the solution is:

$$y(t) = Ae^{-at} + e^{-at}\int e^{at'} b(t')\,dt' \tag{71}$$

so using out differential equation and plugging in a and $b(t)$, we get:

$$i(t) = Ae^{-\frac{1}{RC}t} + e^{-\frac{1}{RC}t}\int e^{\frac{1}{RC}t'}\frac{400}{R}\cos(200t')\,dt' \tag{72}$$

$$i(t) = Ae^{-\frac{1}{RC}t} + \frac{400}{R}e^{-\frac{1}{RC}t}\int e^{\frac{1}{RC}t'}\cos(200t')\,dt' \tag{73}$$

From here, we can apply Equation 70 from the hint:

$$i(t) = Ae^{-\frac{1}{RC}t} + \frac{400}{R}e^{-\frac{1}{RC}t}\frac{1}{\frac{1}{RC}^2+200^2}e^{\frac{1}{RC}t}\left(200\sin(200t) - \frac{1}{RC}\cos(200t)\right) \tag{74}$$

$$i(t) = Ae^{-\frac{1}{RC}t} + \frac{400}{R\left(\frac{1}{RC}^2+200^2\right)}\left(200\sin(200t) - \frac{1}{RC}\cos(200t)\right) \tag{75}$$

Substituting values for $R = 5\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$:

$$i(t) = Ae^{-\frac{1}{5\times10^{-3}}t} + \frac{400}{5\times10^3\left(\frac{1}{5\times10^{-3}}^2+200^2\right)}\left(200\sin(200t) + \frac{1}{5\times10^{-3}}\cos(200t)\right) \tag{76}$$

$$i(t) = Ae^{-200t} + \frac{80\times10^{-3}}{8\times10^4}(200\sin(200t) + 200\cos(200t)) \tag{77}$$

$$i(t) = Ae^{-200t} + 10^{-6}(200\sin(200t) + 200\cos(200t))\text{A} \tag{78}$$

$$i(t) = Ae^{-200t} + 2\times10^{-4}(\sin(200t) + \cos(200t))\text{A} \tag{79}$$

Lastly, we need to solve for the value of the constant A which we will use the initial condition for.

$$i(0) = Ae^{-200\times0} + 2\times10^{-4}(\sin(200\times0) + \cos(200\times0)) \tag{80}$$

$$-2\times10^{-4} = A + 2\times10^{-4} \tag{81}$$

$$A = -4\times10^{-4} \tag{82}$$

Putting this all together, we get that our final solution for current $i(t)$ is:

$$i(t) = -4\times10^{-4}e^{-200t} + 2\times10^{-4}(\sin(200t) + \cos(200t))\text{A} \tag{83}$$

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$$i(t) = -400\mathrm{e}^{-200t} + 200\sin(200t) + 200\cos(200t)\mu\mathrm{A} \quad (84)$$

where the transient response is $-400\mathrm{e}^{-200t}$ (goes to 0 over time) and the forced response is $200\sin(200t) + 200\cos(200t)$.

(d) **(OPTIONAL) Solve for the voltage $v_C(t)$ across the capacitor.**

Solution: There are two ways to go about solving for $v_C(t)$:

(1) You can solve for the voltage by either setting up a differential equation for $v_c(t)$ and solving the differential equation or

(2) use the solution from part (a) and the IV relationship for a capacitor. For this problem, we are simply going to use our solution from part (a) and plug it into the voltage of a capacitor in terms of current:

$$v_C(t) = \frac{1}{C}\int_0^t i(t')\,\mathrm{d}t' + v_C(0) \quad (85)$$

$$v_C(t) = \frac{1}{10{-}6}\int_0^t \left(-4\times10^{-4}\mathrm{e}^{-200t'} + 2\times10^{-4}\sin(200t') + 2\times10^{-4}\cos(200t')\right)\mathrm{d}t' + 1 \quad (86)$$

$$v_C(t) = \frac{1}{10^{-6}}\left(\int_0^t -4\times10^{-4}\mathrm{e}^{-200t'}\,\mathrm{d}t' + \int_0^t 2\times10^{-4}\sin(200t')\,\mathrm{d}t' + \int_0^t 2\times10^{-4}\cos(200t')\,\mathrm{d}t'\right) + 1 \quad (87)$$

$$v_C(t) = -400\left[\frac{e^{-200t'}}{200}\right]_0^t - 200\left[\frac{\cos(200t')}{200}\right]_0^t + 200\left[\frac{\sin(200t')}{200}\right]_0^t + 1 \quad (88)$$

$$v_C(t) = -2e^{-200t} + 2 - (\cos(200t) - 1) + (\sin(200t) - 0) + 1 \quad (89)$$

$$v_C(t) = -2e^{-200t} - \cos(200t) + \sin(200t) + 4\,\mathrm{V} \quad (90)$$

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