Discussion 3A

The following notes and sections are useful for this discussion: Note j, Sections 4.5, 5.1, 5.2.

1. Analyzing a Second-Order Circuit (Hambley Example 4.5)

A DC source is connected to a series RLC circuit by a switch that closes at $t = 0$ as shown in Figure 1. The initial conditions are $i(0) = 0$ and $v_C(0) = 0$.

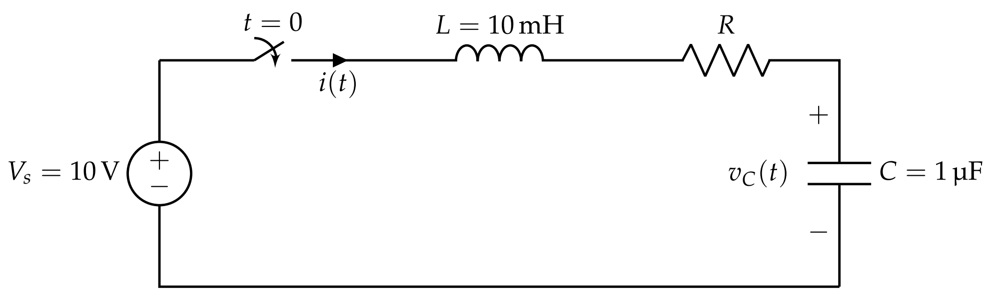


Figure 1: RLC Circuit

(a) **Write the differential equation for $v_c(t)$**

Solution: First, we can write an express for the current in terms of the voltage acorss the capacitance:

$$i(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \tag{1}$$

Then, writing a KVL equation for the circuit, we have:

$$v_L(t) + v_R(t) + v_C(t) = V_s \tag{2}$$

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) + v_C(t) = V_s \tag{3}$$

$$\tag{4}$$

Substituting in the expression for current $i(t)$, we get:

$$LC\frac{\mathrm{d}^2v_C(t)}{\mathrm{d}t^2} + RC\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t) = V_s \tag{5}$$

$$\frac{\mathrm{d}^2v_C(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \frac{1}{LC}v_C(t) = \frac{V_s}{LC} \tag{6}$$

(b) **Solve for $v_C(t)$ if $R = 300\,\Omega$.**

Solution: Our equation is in the form:

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + 2\alpha\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega_0^2x(t) = f(t) \tag{7}$$

where $f(t) = \frac{V_s}{LC}$ which is a constant. Thus, we know that our solution for $v_C(t)$ will be a combination of the particular solution $v_{Cp}(t)$ and the complementary solution $v_{Cc}(t)$.

Since we have a DC source, we know that the transient or complementary solution will go to 0 over time. Thus, our current and voltage are steady/constant and we can replace inductors with short circuits and capcitors with open circuits. This leads us to determine that $v_{Cp}(t) = V_s = 10\,\mathrm{V}$.

Next, we will find the complementary solution $v_{Cc}(t)$ or the homogeneous solution of our differential equation. When finding the complementary solution, we will follow the following 3 steps:

i. Determine the damping ratio and roots of the characteristic equation
ii. Select the appropriate form for the homogeneous solution, dpending on the value of the damping ratio
iii. Add the homogeneous solution to the particular solution and determine the values of the coefficients (K_1 and K_2) based on initial conditions.

Here, we have $R = 300\,\Omega$, so

$$\alpha = \frac{R}{2L} = 1.5 \times 10^4 \tag{8}$$

and the damping ratio $\zeta = \frac{\alpha}{\omega_0} = 1.5$. Since $\zeta > 1$, we have an overdamped case. Solving for the roots of our characteristic equaiton we have:

$$\begin{aligned} s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} && (9) \\ &= -1.5 \times 10^4 + \sqrt{(1.5 \times 10^4)^2 - (10^4)^2} && (10) \\ &= -0.3820 \times 10^4 && (11) \end{aligned}$$

and

$$\begin{aligned} s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} && (12) \\ &= -1.5 \times 10^4 - \sqrt{(1.5 \times 10^4)^2 - (10^4)^2} && (13) \\ &= -2.618 \times 10^4 && (14) \end{aligned}$$

We know that the homogeneous solution has the form $K_1 e^{s_1 t} + K_2 e^{s_2 t}$ leading us to have the general solution:

$$v_C(t) = v_{Cp}(t) + v_{Cc}(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{15}$$

Now, we will find the values of K_1 and K_2 using the given initial conditions. It is given $v_C(0) = 0\,\mathrm{V}$. This gives us that:

$$10 + K_1 + K_2 = 0 \tag{16}$$

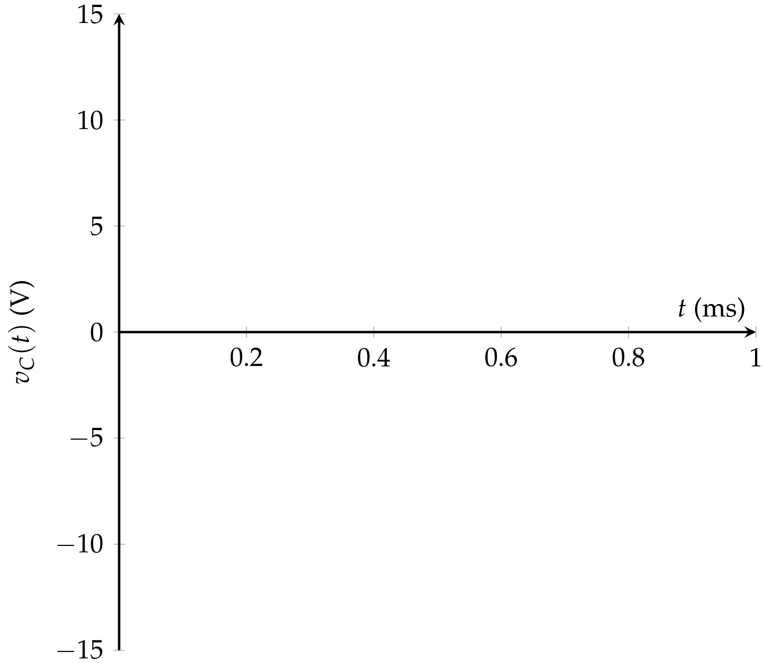
Furthermore, since $i(0) = 0\,\mathrm{A}$ we also know that $i(0) = C\frac{\mathrm{d}v_C(0)}{\mathrm{d}t}$ and thus $\frac{\mathrm{d}v_C(0)}{\mathrm{d}t} = 0$. Taking the derivative of Equation 15 and plugging in $t = 0$, we get

$$\begin{aligned} s_1 K_1 e^{s_1(0)} + s_2 K_2 e^{s_2(0)} &= 0 && (17) \\ s_1 K_1 + s_2 K_2 &= 0 && (18) \end{aligned}$$

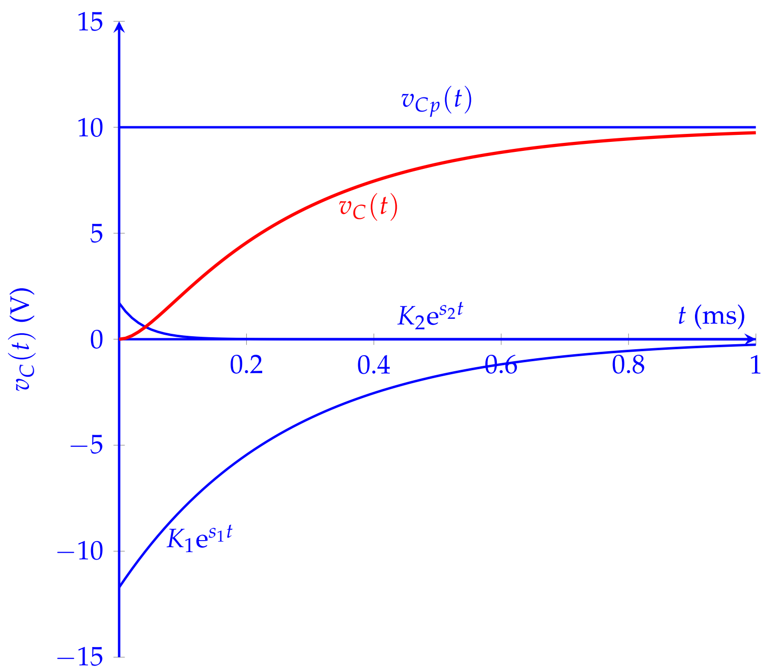
Now, solving the systems of equations, we get that $K_1 = -11.708$ and $K_2 = 1.708$. Substituting these values into Equation 15, we get our final solution:

$$v_C(t) = 10 - 11.708 e^{s_1 t} + 1.708 e^{s_2 t} \tag{19}$$

(c) **Plot the equation you calculated for $v_C(t)$.** It may be helpful to draw out each term in your general solution and then add them together.



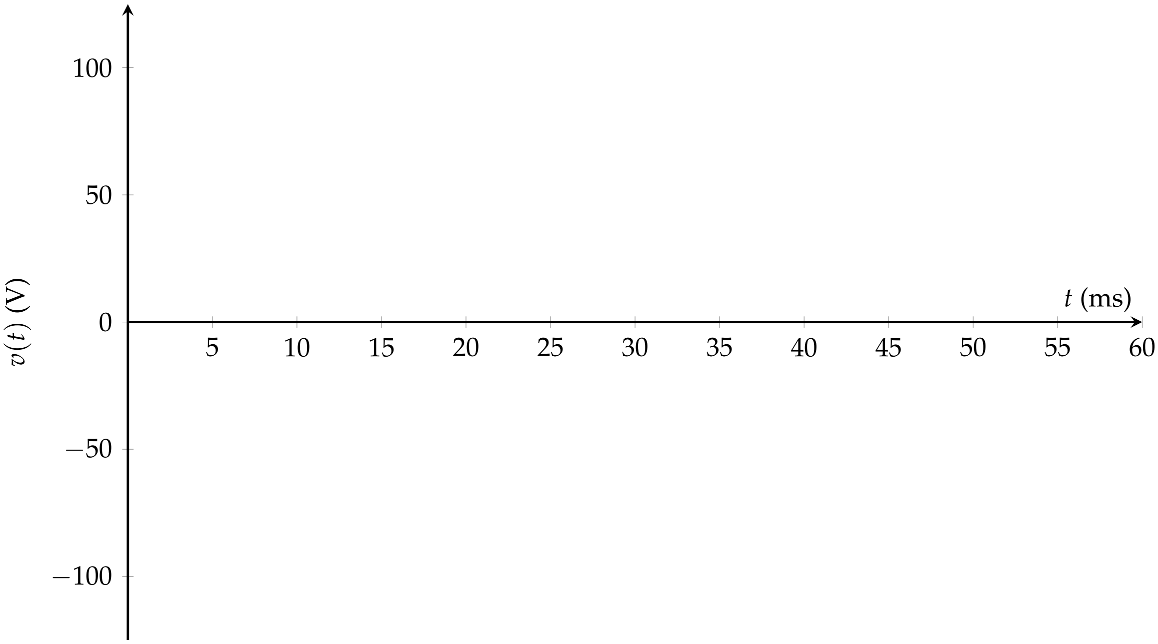
Solution:



2. Power Delivered by a Sinusoidal Source (Hambley Example 5.1)

Suppose that a voltage given by $v(t) = 100\cos(100\pi t)$V is applied to a 50-Ω resistance.

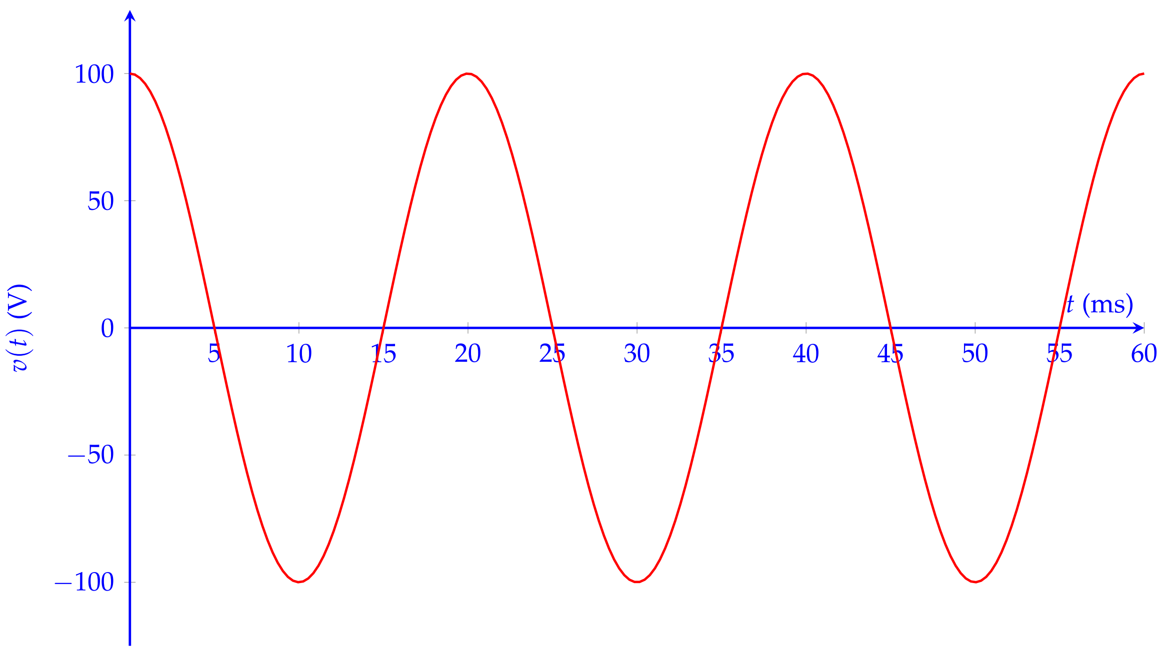
(a) **Sketch $v(t)$ to scale versus time.**



Solution: Comparing the expression given for $v(t)$ to the following equation:

$$v(t) = V_m \cos(\omega t + \theta) \tag{20}$$

we see that $\omega = 100\pi$. Solving for frequency, we get $f = \frac{\omega}{2\pi} = 50\,\text{Hz}$ and that the period is $T = \frac{1}{f} = 20\,\text{ms}$. Using this along with the fact that the peak voltage is 100, we can proceed to plot our sinusoid.



(b) **Find the rms value of the voltage and the average power delivered to the resistance.**

Solution: The definition of V_{rms} is

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t')\,\mathrm{d}t'} \tag{21}$$

but for a sinusoid in the form of $v(t) = V_m \cos(\omega t + \theta)$, the V_{rms} is simply

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \tag{22}$$

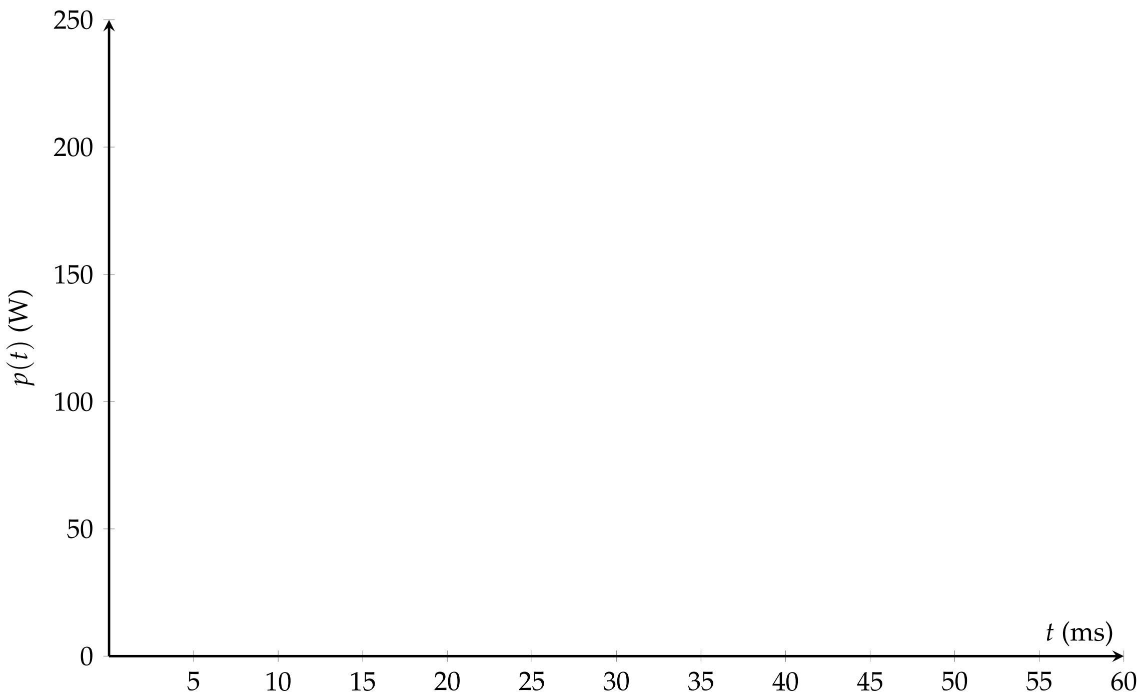
The peak value of the voltage is $V_m = 100V$. Thus, the rms value of

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} = 70.71\,\text{V} \tag{23}$$

Therefore, the average power is:

$$P_{\text{avg}} = \frac{V_{\text{rms}}^2}{R} = \frac{70.71^2}{50} = 100\,\text{W} \tag{24}$$

(c) **Find the power as a function of time and sketch to scale.**

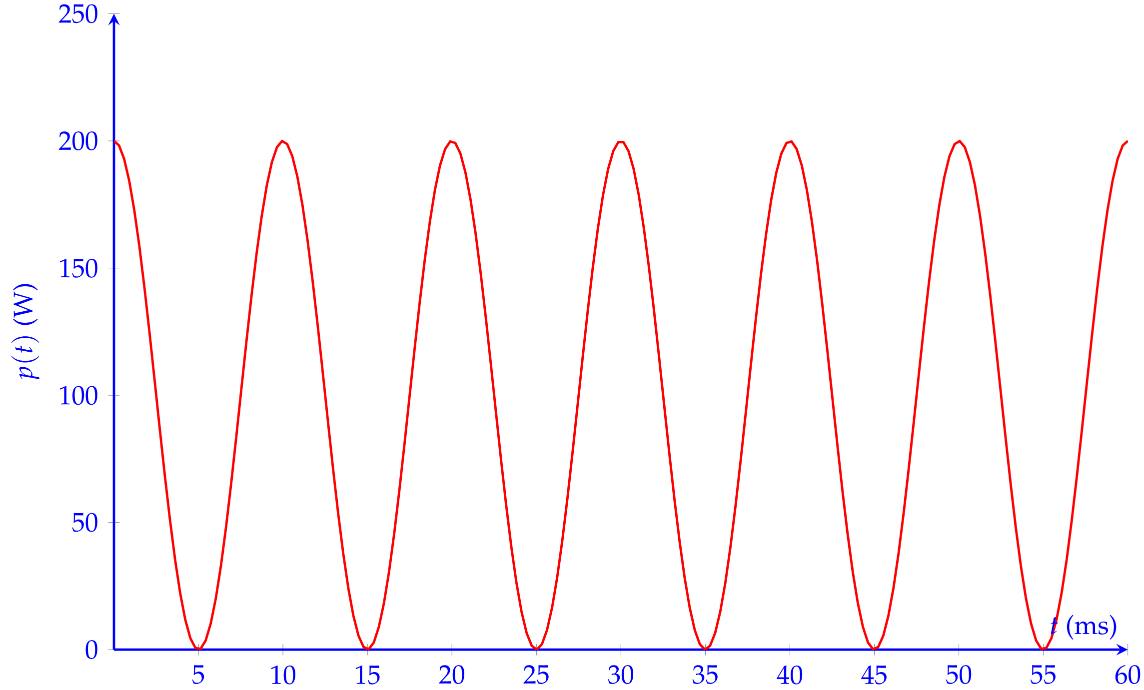


Solution: The power as a function of time is given by:

$$p(t) = \frac{v^2(t)}{R} \tag{25}$$

$$= \frac{(100\cos(100\pi t))^2}{50} \tag{26}$$

$$= 200\cos^2(100\pi t)\text{W} \tag{27}$$


250
200
150
100
50
0
$p(t)$ (W)
5
10
15
20
25
30
35
40
45
50
55
60
t (ms)

3. Complex Number Review

Recall Euler's Formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{28}$$
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \tag{29}$$

Complex exponentials will also be seen written in a magnitude + phase notation as seen in the textbook. The following expressions are equivalent:

$$Re^{j\theta} = R\angle\theta \tag{30}$$

(a) **Convert the following complex numbers from polar form to Cartesian form.**

$$\mathbf{X_1} = 2\sqrt{2}e^{j225^\circ} = 2\sqrt{2}\angle 225^\circ \tag{31}$$
$$\mathbf{X_2} = e^{-j90^\circ} = 1\angle -90^\circ \tag{32}$$

Solution: Using Euler's formula, we have that:

$$\mathbf{X_1} = 2\sqrt{2}\cos(225^\circ) + j2\sqrt{2}\sin(225^\circ) \tag{33}$$
$$= -2\sqrt{2}\frac{\sqrt{2}}{2} - j2\sqrt{2}\frac{\sqrt{2}}{2} \tag{34}$$
$$= -2 - 2j \tag{35}$$

Similarly for $\mathbf{X_2}$:

$$\mathbf{X_2} = \cos(-90^\circ) + j\sin(-90^\circ) \tag{36}$$
$$= 0 + j(-1) \tag{37}$$
$$= -j \tag{38}$$

(b) **Convert the following complex numbers from Cartesian form to polar form.**

$$\mathbf{X_3} = 3 + j4 \tag{39}$$
$$\mathbf{X_4} = -\sqrt{3} + j \tag{40}$$

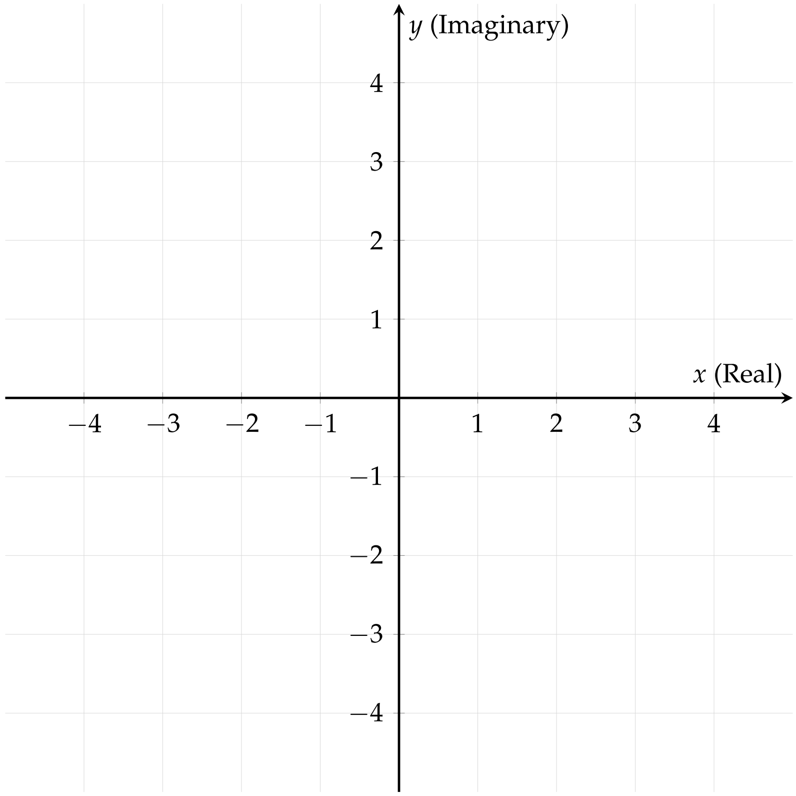
Solution: The magnitude of $|\mathbf{X_3}| = \sqrt{3^2 + 4^2} = 5$. The phase is $\theta = \arctan\frac{4}{3} = 50.13^\circ$. Therefore,

$$\mathbf{X_3} = 5e^{j50.13^\circ} = 5\angle 50.13^\circ \tag{41}$$

The magnitude of $|\mathbf{X_4}| = \sqrt{\sqrt{3}^2 + 1^2} = 2$. The phase $\arctan\frac{1}{-\sqrt{3}} = -30^\circ$, which accounting for the quadrant, will end up being $\theta = 150^\circ$. Therefore,

$$\mathbf{X_4} = 2e^{j150^\circ} = 2\angle 150^\circ \tag{42}$$

(c) **Plot the vectors from parts (a) and (b) on the complex plane.**



Solution: