Discussion 3B

1. Adding Sinusoids with Phasors (Hambley Example 5.3)

Suppose you are given two sinusoidal inputs:

$$v_1(t) = 20\cos(\omega t - 45^\circ) \tag{1}$$
$$v_2(t) = 10\sin(\omega t + 60^\circ) \tag{2}$$

Add together the inputs $v_s(t) = v_1(t) + v_2(t)$ such that $v_s(t)$ is composed of a single cosine term.

Solution: Let's start by converting $v_2(t)$ into a cosine function. Using the relationship:

$$\sin(\omega t) = \cos(\omega t - 90^\circ) \tag{3}$$

We have that:

$$v_2(t) = 10\cos(\omega t - 30^\circ) \tag{4}$$

The phasors corresponding to $v_1(t)$ and $v_2(t)$ are:

$$\mathbf{V_1} = 20\angle{-45^\circ} \tag{5}$$
$$\mathbf{V_2} = 10\angle{-30^\circ} \tag{6}$$

Now we will use the complex-number representation of phasors to add the input signals together:

$$\mathbf{V_s} = \mathbf{V_1} + \mathbf{V_2} \tag{7}$$
$$= 20\angle{-45^\circ} + 10\angle{-30^\circ} \tag{8}$$
$$= 20\cos(-45^\circ) + j20\sin(-45^\circ) + 10\cos(-30^\circ) + j10\sin(-30^\circ) \tag{9}$$
$$= 14.14 - j14.14 + 9.8660 - j5 \tag{10}$$
$$= 22.80 - j19.14 \tag{11}$$
$$= 29.77\angle{-40.01^\circ} \tag{12}$$

Converting our phasor $\mathbf{V_s}$ back into a sinusoid, we get

$$v_s(t) = 29.77\cos(\omega t - 40.01^\circ) \tag{13}$$

2. Inductor Impedance (Hambley Exercise 5.6)

(a) A sample inductor circuit is given in Figure 1.

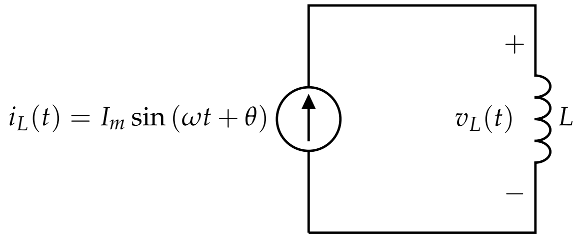


Figure 1: Sample inductor circuit

Derive the impedance of the inductor Z_L given that $\mathbf{V_L} = Z_L \mathbf{I_L}$.

Solution: Using the equation for an inductor, we can solve for the sinusoidal representation of $v_L(t)$:

$$v_L(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} \tag{14}$$

$$v_L(t) = L\frac{\mathrm{d}}{\mathrm{d}t}\left(I_m \sin(\omega t + \theta)\right) \tag{15}$$

$$v_L(t) = LI_m\omega \cos(\omega t + \theta) \tag{16}$$

Rewriting our voltage and current as phasors in polar form we have that:

$$\mathbf{V_L} = LI_m\omega\angle\theta = LI_m\omega e^{j\theta} \tag{17}$$

$$\mathbf{I_L} = I_m\angle\theta - 90^\circ = I_m e^{j(\theta - 90^\circ)} \tag{18}$$

We can then write $\mathbf{V_L}$ to be in terms of $\mathbf{I_L}$ which then allows us to find impedance for the inductor:

$$\begin{aligned}
\mathbf{V_L} &= LI_m\omega e^{j(\theta+90^\circ-90^\circ)} && (19)\\
&= LI_m\omega e^{j(\theta-90^\circ)}e^{j90^\circ} && (20)\\
&= L\omega\angle 90^\circ\left(I_m\angle\theta - 90^\circ\right) && (21)\\
&= L\omega\angle 90^\circ \mathbf{I_L} && (22)\\
\mathbf{V_L} &= j\omega L \mathbf{I_L} && (23)
\end{aligned}$$

Therefore we have that $Z_L = j\omega L$.

(b) Assume that now you are told that a voltage of $v_L(t) = 10\cos(20t)$ is applied to a 0.25-H inductance.

Calculate the impedance of the inductor, the phasor current, and the phasor voltage.

Solution: From the previous part we have the $Z_L = j\omega L$. Plugging in $\omega = 200$ and $L = 0.25\,\mathrm{H}$, we get:

$$Z_L = j(20)(0.25) = j5 \tag{24}$$

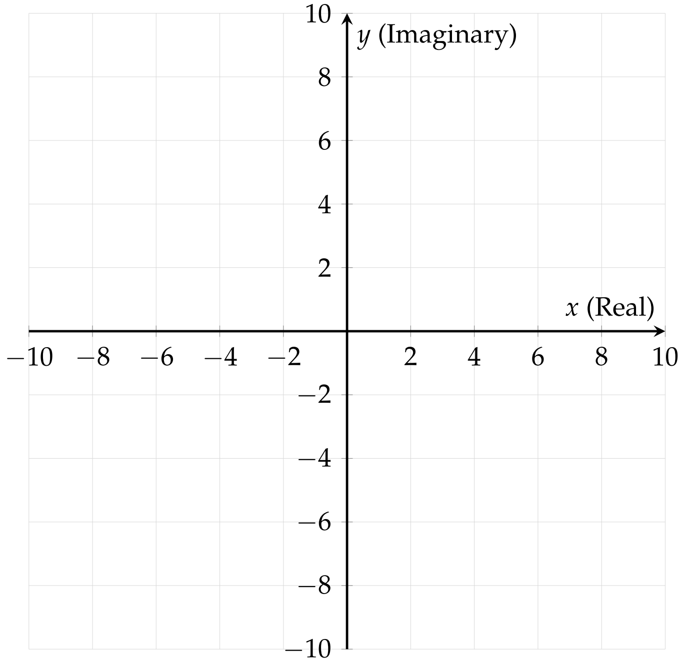
For the phasor voltage, we have that the peak voltage is 10 and that there is no phase. Therefore, the phasor voltage is

$$\mathbf{V_L} = 10\angle 0^\circ = 10 + j0 \tag{25}$$

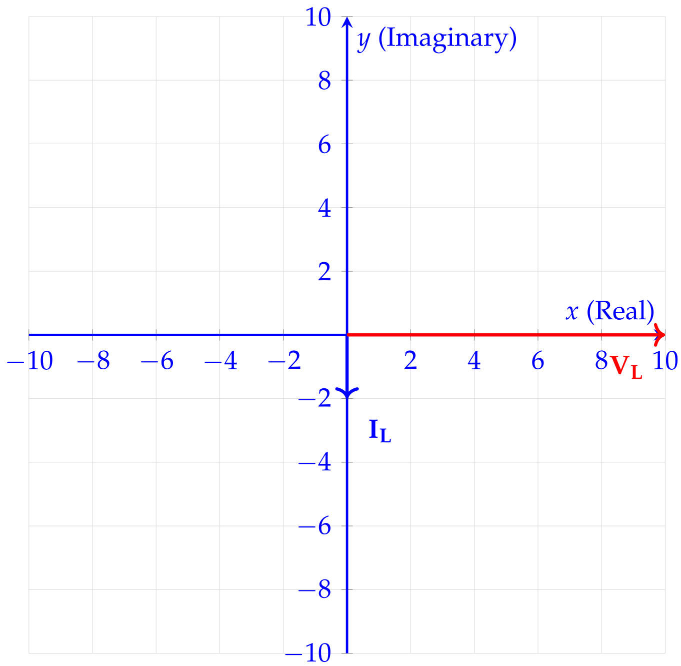
For the phasor current, we can use the fact that impedance, current, and voltage are related via: $\mathbf{V_L} = Z_L \mathbf{I_L}$. Thus, solving for $\mathbf{I_L}$ we have

$$\mathbf{I_L} = \frac{\mathbf{V_L}}{Z_L} = \frac{10\angle 0^\circ}{5\angle 90^\circ} = 2\angle -90^\circ = -2j \tag{26}$$

(c) **Sketch the phasors $\mathbf{V_L}$ and $\mathbf{I_L}$ on the complex plane and state the phase relationship of the current and voltage of a pure inductance.**



Solution:



Notice that the current "lags" behind the voltage of the inductor by $90°$.

3. Series and Parallel Combinations of Complex Impedances

Consider the circuit shown in Figure 2.

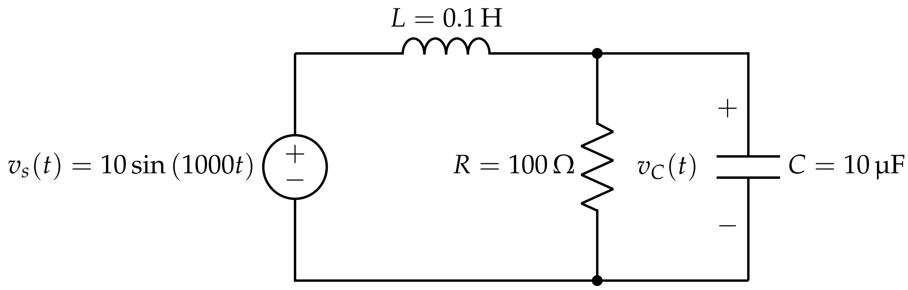


Figure 2: RLC Circuit

(a) **Find the voltage $v_C(t)$ in steady state.**

Solution: The phasor for voltage source is $\mathbf{V_s} = 10\angle{-90^\circ}$. Notice that $v_s(t)$ is a sine function which is why the phase has 90° subtracted from it. The angular frequency of the source is $\omega = 1000$.

The impedance of the inductor is:

$$Z_L = j\omega L = j1000 \times 0.1 = j100\,\Omega \tag{27}$$

And the impedance of the capacitor is:

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{1000 \times 10 \times 10^{-6}} = -j100\,\Omega \tag{28}$$

To find $\mathbf{V_C}$, we need to combine the resistance and impedance of the capacitor in parallel to find the equivalent impedance across the voltage. Then, using the voltage divider principle to compute the voltage acros the RC combination. The impedance of the parallel RC circuit is:

$$Z_{RC} = \frac{1}{\frac{1}{R} + \frac{1}{Z_C}} \tag{29}$$

$$= \frac{1}{\frac{1}{100} + \frac{1}{-j100}} \tag{30}$$

$$= \frac{1}{0.01 + j0.01} \tag{31}$$

$$= \frac{1\angle 0^\circ}{0.01414\angle 45^\circ} \tag{32}$$

$$= 70.71\angle{-45^\circ} \tag{33}$$

In Cartesian form, we have:

$$Z_{RC} = 50 - j50 \tag{34}$$

Now, using the voltage divider principle with impedances, we have:

$$\mathbf{V_C} = \frac{Z_{RC}}{Z_L + Z_{RC}}\mathbf{V_s} \tag{35}$$

$$= 10\angle{-90^\circ}\frac{70.71\angle{-45^\circ}}{j100 + 50 - j50} \tag{36}$$

$$= 10\angle{-90^\circ}\frac{70.71\angle{-45^\circ}}{50 + j50} \tag{37}$$

$$= 10\angle{-90^\circ}\frac{70.71\angle{-45^\circ}}{70.71\angle 45^\circ} \tag{38}$$

$$= 10\angle{-180^\circ} \tag{39}$$

Converting the phasor to a time function, we have

$$v_C(t) = 10\cos\left(1000t - 180^\circ\right) = -10\cos\left(1000t\right) \tag{40}$$

(b) **Find the phasor current through each element.**

Solution: First computing the overall current,

$$\mathbf{I} = \frac{\mathbf{V_s}}{Z_L + Z_{RC}} \tag{41}$$

$$= \frac{10\angle{-90^\circ}}{j100 + 50 - j50} \tag{42}$$

$$= \frac{10\angle{-90^\circ}}{70.71\angle 45^\circ} \tag{43}$$

$$= 0.1414\angle{-135^\circ} \tag{44}$$

Now for the element currents, we have:

$$\mathbf{I_R} = \frac{\mathbf{V_C}}{R} = \frac{10\angle{-180^\circ}}{100} = 0.1\angle{-180^\circ} \tag{45}$$

$$\mathbf{I_C} = \frac{\mathbf{V_C}}{Z_C} = \frac{10\angle{-180^\circ}}{100\angle{-90^\circ}} = 0.1\angle{-90^\circ} \tag{46}$$

(c) **Sketch a phasor diagram showing the currents and the source voltage.**

Solution: