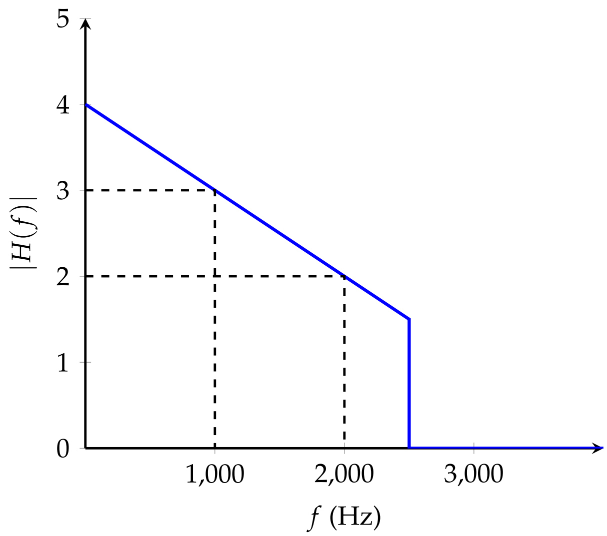
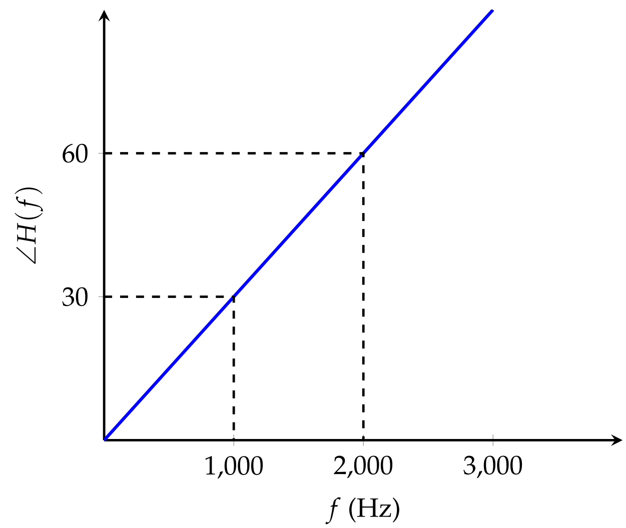
Discussion 4B

1. Using the Transfer Function to Determine the Output (Hambley Example 6.1)

The transfer function $H(f)$ of a filter is shown in Figure 1.



(a) Transfer Function Magnitude



(b) Transfer Function Phase

Figure 1: Transfer Function $H(f)$

If the input signal is given by

$$v_{\text{in}}(t) = 2\cos(2000\pi t + 40^\circ) + 2\cos(4000\pi t) \tag{1}$$

find an expression for the output of the filter $v_{\text{out}}(t)$.

Solution: Since our input is composed on two sinusoids with different frequencies we need to analyze them separately. Let's call:

$$v_{\text{in},1} = 2\cos(2000\pi t + 40^\circ) \tag{2}$$

$$v_{\text{in},2} = 2\cos(4000\pi t) \tag{3}$$

Let's first analyze the output of $v_{\text{in},1}$. By inspection, we see that $\omega = 2000\pi$ and therefore $f = \frac{\omega}{2\pi} = 1000\,\text{Hz}$. Using the provided graphs of the magnitude and phase, we can determine $|H(1000)| = 3$ and $\angle H(1000) = 30^\circ$. Putting this together we have:

$$H(1000) = 3\angle 30^\circ = \frac{\mathbf{V}_{\text{out},1}}{\mathbf{V}_{\text{in},1}} \tag{4}$$

The phasor for the input signal is $\mathbf{V}_{\text{in},1} = 2\angle 40^\circ$, so solving for the output phasor we have:

$$\mathbf{V}_{\text{out},1} = H(1000)\mathbf{V}_{\text{in},1} \tag{5}$$

$$= 3\angle 30^\circ \times 2\angle 40^\circ \tag{6}$$

$$= 6\angle 70^\circ \tag{7}$$

Converting the output phasor back into a time function, we have:

$$v_{\text{out},1}(t) = 6\cos\left(2000\pi t + 70^\circ\right) \tag{8}$$

Now, we will apply the same process for $v_{\text{in},2}$. We observe that $\omega = 4000\pi$ and therefore, $f = 2000\,\text{Hz}$. Then using the graphs we know that $|H(2000)| = 2$ and $\angle H(2000) = 60$. We also can represent $v_{\text{in},2}(t)$ as a phasor which would be $2\angle 0^\circ$. Putting this together and solving for the output phasor,

$$\mathbf{V}_{\text{out},2} = H(2000)\mathbf{V}_{\text{in},2} \tag{9}$$

$$= 2\angle 60^\circ \times 2\angle 0^\circ \tag{10}$$

$$= 4\angle 60^\circ \tag{11}$$

Converting this into a time function, we get:

$$v_{\text{out},2}(t) = 4\cos\left(4000\pi t + 60^\circ\right) \tag{12}$$

Now combining the two output sinusoids we get:

$$v_{\text{out}}(t) = 6\cos\left(2000\pi t + 70^\circ\right) + 4\cos\left(4000\pi t + 60^\circ\right) \tag{13}$$

2. ***RC* Filter (Hambley Example 6.3)**

Suppose you have the RC circuit shown in Figure 2.

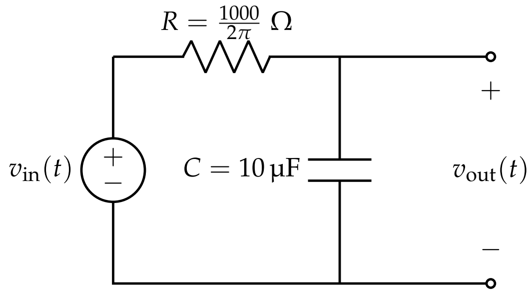


Figure 2: RC Lowpass Circuit

(a) **Determine the transfer function $H(f)$ of the given circuit. Then, classify what type of filter this circuit is.** Recall that the transfer function $H(f)$ is defined as the ratio of the output phasor to the input phasor.

$$H(f) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} \tag{14}$$

Solution: To determine the transfer function, we can apply a sinusoidal input signal with phasor $\mathbf{V}_{\text{in}}$ and then analyze the behavior of the circuit as a function of the source frequence f. The phasor current is given by dividing the input voltage phasor by the complex impedance of the circuit.

$$\mathbf{I} = \frac{\mathbf{V}_{\text{in}}}{R + \frac{1}{j2\pi f C}} \tag{15}$$

The phasor for the output voltage would be the product of the phasor current and the impedance of the capacitor:

$$\mathbf{V}_{\text{out}} = \frac{1}{j2\pi f C}\mathbf{I} \tag{16}$$

Then, substituting in our expression for $\mathbf{I}$, we have:

$$\mathbf{V}_{\text{out}} = \frac{1}{j2\pi f C} \times \frac{\mathbf{V}_{\text{in}}}{R + \frac{1}{j2\pi f C}} \tag{17}$$

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1}{j2\pi f C\left(R + \frac{1}{j2\pi f C}\right)} \tag{18}$$

$$H(f) = \frac{1}{1 + j2\pi f RC} \tag{19}$$

An alternate solution involves applying the voltage divider principle with the complex impedances of the circuit.

$$\mathbf{V}_{\text{out}} = \frac{Z_C}{Z_R + Z_C}\mathbf{V}_{\text{in}} \tag{20}$$

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{\frac{1}{j2\pi f C}}{R + \frac{1}{j2\pi f C}} \tag{21}$$

$$H(f) = \frac{1}{1 + j2\pi f RC} \tag{22}$$

Plugging in the values for R and C, we get that our transfer function is:

$$H(f) = \frac{1}{1 + \mathrm{j}\frac{f}{2\pi \times \frac{1000}{2\pi} \times 10 \times 10^{-6}}} = \frac{1}{1 + \mathrm{j}\frac{f}{100}} \tag{23}$$

Now, to determine what type of filter this is, we can analyze the behavior for low and high ω. As $\omega \to 0$, notice that the magnitude of the transfer function $H(f)$ will get closer and closer to 1. On the other hand as $\omega \to \infty$, we see that the magnitude of the transfer function goes to 0. Therefore, we have the our circuit will attentuate/reduce high frequency and maintain low frequency signals, which leads us to conclude that we have a lowpass RC filter.

(b) Suppose, that you are given:

$$v_{\text{in}}(t) = 5\cos(20\pi t) + 5\cos(200\pi t) + 5\cos(2000\pi t) \tag{24}$$

Find an expression for the output signal $v_{\text{out}}(t)$.

Solution: Let's call $v_{\text{in},1} = 5\cos(20\pi t)$, $v_{\text{in},2} = 5\cos(200\pi t)$, and $v_{\text{in},3} = 5\cos(2000\pi t)$ and their phasors $\mathbf{V}_{\text{in},1}$, $\mathbf{V}_{\text{in},2}$, and $\mathbf{V}_{\text{in},3}$, respectively.

We have that:

$$\mathbf{V}_{\text{in},1} = 5\angle 0^\circ \tag{25}$$

$$\mathbf{V}_{\text{in},2} = 5\angle 0^\circ \tag{26}$$

$$\mathbf{V}_{\text{in},3} = 5\angle 0^\circ \tag{27}$$

Now, we must analyze the value of the transfer function at each of the different input frequency values. Recall that $f = \frac{\omega}{2\pi}$, therefore $f_1 = 10\,\text{Hz}$, $f_2 = 100\,\text{Hz}$, and $f_3 = 1000\,\text{Hz}$.

$$H(f_1) = H(10) = \frac{1}{1 + \mathrm{j}\frac{10}{100}} = 0.9950\angle{-5.71^\circ} \tag{28}$$

$$H(f_2) = H(100) = \frac{1}{1 + \mathrm{j}\frac{100}{100}} = 0.7071\angle{-45^\circ} \tag{29}$$

$$H(f_3) = H(1000) = \frac{1}{1 + \mathrm{j}\frac{1000}{100}} = 0.0995\angle{-84.29^\circ} \tag{30}$$

Recall, that to simplify a fraction of complex numbers we can convert them into their polar form, then divide the magnitudes and subtract their phases. Taking $H(f_1)$ as an example:

$$H(f_1) = \frac{1}{1 + \mathrm{j}0.1} \tag{31}$$

$$= \frac{1\angle 0^\circ}{\sqrt{1^2 + 0.1^2}\angle \arctan\left(\frac{0.1}{1}\right)} \tag{32}$$

$$= \frac{1\angle 0^\circ}{1.005\angle 5.71^\circ} \tag{33}$$

$$= \frac{1}{1.005}\angle 0^\circ - 5.71^\circ \tag{34}$$

$$= 0.9950\angle{-5.71^\circ} \tag{35}$$

The output phasor for each input phasor is simply the product of the input phasor with the transfer function evaluated at the given frequency $\mathbf{V}_{\text{out},1} = H(f_1)\mathbf{V}_{\text{in},1}$. Thus we have:

$$\mathbf{V}_{\text{out},1} = H(f_1)\mathbf{V}_{\text{in},1} = (0.9950\angle -5.71^\circ) \times (5\angle 0^\circ) \tag{36}$$
$$= 4.975\angle -5.71^\circ \tag{37}$$
$$\mathbf{V}_{\text{out},2} = H(f_2)\mathbf{V}_{\text{in},2} = 0.7071\angle -45^\circ \times (5\angle 0^\circ) \tag{38}$$
$$= 3.535\angle -45^\circ \tag{39}$$
$$\mathbf{V}_{\text{out},3} = H(f_3)\mathbf{V}_{\text{in},3} = 0.0995\angle -84.29^\circ \times (5\angle 0^\circ) \tag{40}$$
$$= 0.4975\angle -84.29^\circ \tag{41}$$

Converting the output phasors into their sinusoidal representations, we get:

$$v_{\text{out},1}(t) = 4.975\cos(20\pi t - 5.71^\circ) \tag{42}$$
$$v_{\text{out},2}(t) = 3.535\cos(200\pi t - 45^\circ) \tag{43}$$
$$v_{\text{out},3}(t) = 0.4975\cos(2000\pi t - 84.29^\circ) \tag{44}$$

Lastly, we we sum together the output components to get:

$$v_{\text{out}} = 4.975\cos(20\pi t - 5.71^\circ) + 3.535\cos(200\pi t - 45^\circ) + 0.4975\cos(2000\pi t - 84.29^\circ) \tag{45}$$

The important concept to understand here is that each input signal is impacted differently by the filter due to their varying frequencies. You can see that the $f = 10\,\text{Hz}$ input signal did not change much in amplitude and phase. On the other hand $f = 100\,\text{Hz}$ had its amplitude reduced by a factor of 0.7071 and phase shifted by -45°. Lastly the $f = 1000\,\text{Hz}$ signal is reduced by a whole order of magnitude (10^{-1}), demonstrating the lowpass behavior of the filter.

3. LR Filter (Hambley Exercise 6.5)

Suppose you are given the following circuit:

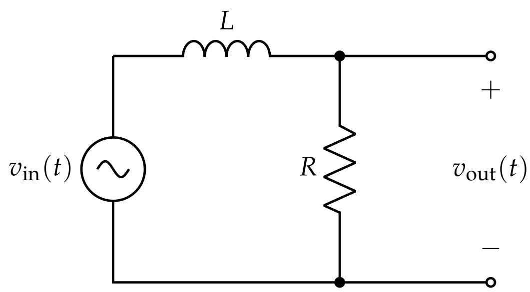


Figure 3: LR Circuit

Derive the transfer function of this filter, classify what type of filter it is, and determine an expression for the half-power frequency f_B.

Solution: Let's first convert our circuit into the phasor domain.

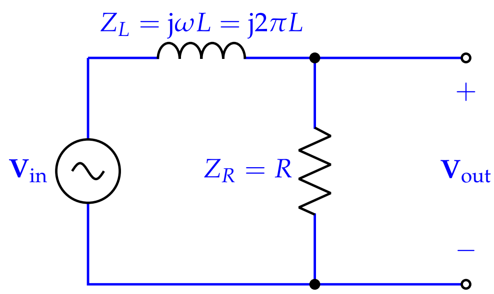


Figure 4: LR Circuit

Now, we can apply the voltage divider principle to write $\mathbf{V}_{out}$ in terms of $\mathbf{V}_{in}$. You can also apply node-voltage analysis to solve for the transfer function as well.

$$\mathbf{V}_{out} = \frac{Z_R}{Z_R + Z_L}\mathbf{V}_{in} \tag{46}$$

$$\mathbf{V}_{out} = \frac{R}{R + j2\pi f L}\mathbf{V}_{in} \tag{47}$$

$$\frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{R}{R + j2\pi f L} \tag{48}$$

$$H(f) = \frac{R}{R + j2\pi f L} \tag{49}$$

$$H(f) = \frac{1}{1 + j2\pi f \frac{L}{R}} \tag{50}$$

Notice that we can rewrite our transfer function to take on the same form as the low pass filter:

$$H(f) = \frac{1}{1 + j\frac{f}{f_B}} = \frac{1}{1 + j\frac{f}{\frac{R}{2\pi L}}} \tag{51}$$

Another way to determine that this is a lowpass filter is to analyze the magnitude of the transfer function as $f \to 0$ and $f \to \infty$. As $f \to 0$, we notice that the magntitude of the filter approaches 1, where as when $f \to \infty$, the magnitude of the denominator will go to ∞, which indicates that the magnitude of the overall transfer function will go to 0. This behavior of maintaining low frequency signals but reducing high frequency signals makes it a lowpass filter.

For this LR filter, we have that the half-power frequency is:

$$f_B = \frac{R}{2\pi L} \tag{52}$$