Discussion 6B

1. Changing Coordinates and Systems of Differential Equations, II

In the previous discussion, we analyzed and solved a pair of differential equations where the variables of interest were coupled (at least one equation depends on more than variable).

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{1}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t). \tag{2}$$

We solved this system by using a coordinate transformation that gave us a decoupled system of equations. In the last discussion we were simply handed this transformation, but in this discussion we will construct the transformation for ourselves.

We will focus our explorations on the voltages across the capacitors in the following circuit.

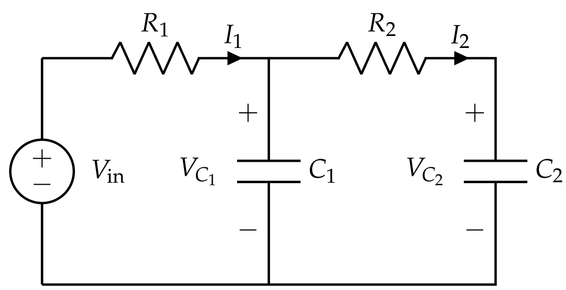

Figure 1: Two dimensional system: a circuit with two capacitors, like the one in lecture.

(a) **Write the system of differential equations governing the voltages across the capacitors** V_{C_1}, V_{C_2}. Use the following values: $C_1 = 1\,\mu\text{F}, C_2 = \frac{1}{3}\mu\text{F}, R_1 = \frac{1}{3}\text{M}\Omega, R_2 = \frac{1}{2}\text{M}\Omega$.

Formulate your system as a matrix differential equation.

(b) Now, for the rest of this problem, we denote $V_{C_1}(t) = z_1(t), V_{C_2}(t) = z_2(t)$. Suppose that V_{in} was at 7 V for a long time, and then transitioned to be 0 V at time $t = 0$. This "new" system of differential equations (valid for $t \geq 0$) is

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{3}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t) \tag{4}$$

with initial conditions $z_1(0) = 7$ and $z_2(0) = 7$.

Write out the differential equations *and initial conditions* in matrix/vector form.

(c) **Find the eigenvalues $\lambda_1,\ \lambda_2$ and eigenspaces for the matrix corresponding to the differential equation matrix above.**

(HINT: Remember how we find λ for a matrix; we solve $\det(A - \lambda I) = 0$.*)*

(d) **Using the eigenvectors from above, change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $y_{\lambda_1}(t), y_{\lambda_2}(t)$.** These variables represent eigenbasis-aligned coordinates.

As a reminder, below is the general strategy we are following to solve this system, and now we've filled in the remaining question from the previous discussion regarding the origin of the transform V.

$$
\begin{tikzcd}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = A_z\vec{z}(t) \arrow[r, squiggly, "\text{Too difficult}"'] \arrow[d, "\substack{\text{Change of}\\\text{variables to get}\\\text{a diagonal system}}"'] & \vec{z}(t) = \ldots \\
\frac{\mathrm{d}}{\mathrm{d}t}\vec{y_\lambda}(t) = A_{y_\lambda}\vec{y_\lambda}(t) \arrow[r, "\substack{\text{Solve a}\\\text{diagonal system}}"'] & \vec{y_\lambda}(t) = \ldots \arrow[u, "\substack{\text{Undo change}\\\text{of variables}}"']
\end{tikzcd}
$$

Figure 2: A Strategy to Solve for $\vec{z}(t)$

(e) **Solve the differential equation for $y_{\lambda_i}(t)$ in the eigenbasis.** *Don't forget about the initial conditions!*

(f) **Convert your solution back into the original coordinates to find $z_i(t)$.**

(g) **(PRACTICE)** In part 1.b of the discussion, we make a simplifying assumption V_{in} transitions from $7\,\text{V}$ to $0\,\text{V}$ at $t = 0$. We now consider the setting, where the voltage V_{in} transitions from $0\,\text{V}$ to $7\,\text{V}$ at $t = 0$, i.e we have $V_{\text{in}}(t) = 7\,\text{V}$ for $t \geq 0$.

Find the solution for $z_i(t)$ under the assumption that $V_{\text{in}}(t) = 7\,\text{V}$ for $t \geq 0$ (that is, our system is now nonhomogeneous).

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