Discussion 6B

1. Changing Coordinates and Systems of Differential Equations, II

In the previous discussion, we analyzed and solved a pair of differential equations where the variables of interest were coupled (at least one equation depends on more than variable).

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{1}$$
$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t). \tag{2}$$

We solved this system by using a coordinate transformation that gave us a decoupled system of equations. In the last discussion we were simply handed this transformation, but in this discussion we will construct the transformation for ourselves.

We will focus our explorations on the voltages across the capacitors in the following circuit.

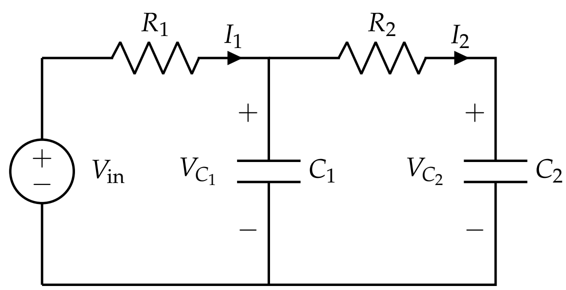

Figure 1: Two dimensional system: a circuit with two capacitors, like the one in lecture.

(a) **Write the system of differential equations governing the voltages across the capacitors** V_{C_1}, V_{C_2}. Use the following values: $C_1 = 1\,\mu\text{F}, C_2 = \frac{1}{3}\mu\text{F}, R_1 = \frac{1}{3}\text{M}\Omega, R_2 = \frac{1}{2}\text{M}\Omega$.

Formulate your system as a matrix differential equation.

Solution: We start by solving for the currents and voltages across the capacitors:

$$V_{C_2} = V_{C_1} - I_2R_2, \quad I_2 = C_2\frac{\mathrm{d}}{\mathrm{d}t}V_{C_2} \tag{3}$$
$$V_{\text{in}} - I_1R_1 = V_{C_1}, \quad I_1 = I_2 + C_1\frac{\mathrm{d}}{\mathrm{d}t}V_{C_1} \tag{4}$$

This yields

$$I_1 = \frac{V_{\text{in}}}{R_1} - \frac{V_{C_1}}{R_1}, \quad I_2 = \frac{V_{C_1}}{R_2} - \frac{V_{C_2}}{R_2} \tag{5}$$

Now, we can plug into the formula for current across a capacitor:

$$\frac{\mathrm{d}V_{C_1}}{\mathrm{d}t} = \frac{1}{C_1}(I_1 - I_2) \tag{6}$$
$$= \frac{1}{C_1}\left(\frac{V_{\text{in}}}{R_1} - \frac{V_{C_1}}{R_1} - \frac{V_{C_1}}{R_2} + \frac{V_{C_2}}{R_2}\right) \tag{7}$$
$$= -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_{C_1} + \frac{V_{C_2}}{R_2C_1} + \frac{V_{\text{in}}}{R_1C_1} \tag{8}$$

$$\frac{\mathrm{d}V_{C_2}}{\mathrm{d}t} = \frac{1}{C_2}(I_2) \tag{9}$$
$$= \frac{V_{C_1}}{R_2C_2} - \frac{V_{C_2}}{R_2C_2} \tag{10}$$

Now, we group the terms into a matrix with the values given above,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{\text{in}}(t) \tag{11}$$

Plugging in the values for R, C yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{\text{in}}(t) \tag{12}$$

(b) Now, for the rest of this problem, we denote $V_{C_1}(t) = z_1(t), V_{C_2}(t) = z_2(t)$. Suppose that V_{in} was at 7 V for a long time, and then transitioned to be 0 V at time $t = 0$. This "new" system of differential equations (valid for $t \geq 0$) is

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{13}$$
$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t) \tag{14}$$

with initial conditions $z_1(0) = 7$ and $z_2(0) = 7$.

Write out the differential equations *and initial conditions* in matrix/vector form.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{15}$$
$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} \tag{16}$$

We will define the differential matrix as A_z, where

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = A_z\vec{z}(t) \tag{17}$$
$$A_z = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \tag{18}$$

(c) **Find the eigenvalues λ_1, λ_2 and eigenspaces for the matrix corresponding to the differential equation matrix above.**

(HINT: Remember how we find λ for a matrix; we solve $\det(A - \lambda I) = 0$.*)*

Solution: Eigenvalues λ and eigenvectors v of matrix A are given by

$$A_z v = \lambda v \tag{19}$$

To find the eigenvalues, we take the determinant:

$$\det(A_z - \lambda I) = 0 \tag{20}$$

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$$\det\left(\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix}\right) = 0 \tag{21}$$

Solving using the 2×2 determinant formula (or by Gaussian elimination):

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc \tag{22}$$

$$(-5-\lambda)(-6-\lambda) - 12 = 0 \tag{23}$$

$$30 + 11\lambda + \lambda^2 - 12 = 0 \tag{24}$$

$$\lambda^2 + 11\lambda + 18 = 0 \tag{25}$$

$$(\lambda + 9)(\lambda + 2) = 0 \tag{26}$$

Giving:

$$\lambda = -9, -2 \tag{27}$$

The eigenspace associated with $\lambda_1 = -9$ is given by:

$$\begin{bmatrix} -5+9 & 2 \\ 6 & -6+9 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{28}$$

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{29}$$

$$\vec{v}_{\lambda_1} = \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ \alpha \in \mathbb{R} \tag{30}$$

The eigenspace associated with $\lambda_2 = -2$ is given by:

$$\begin{bmatrix} -5+2 & 2 \\ 6 & -6+2 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{32}$$

$$\vec{v}_{\lambda_2} = \beta \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ \beta \in \mathbb{R} \tag{33}$$

(d) **Using the eigenvectors from above, change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $y_{\lambda_1}(t), y_{\lambda_2}(t)$.** These variables represent eigenbasis-aligned coordinates.

As a reminder, below is the general strategy we are following to solve this system, and now we've filled in the remaining question from the previous discussion regarding the origin of the transform V.

$$\begin{tikzcd}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = A_z\vec{z}(t) \arrow[r, squiggly, "\text{Too difficult}"'] \arrow[d, "\text{Change of variables to get a diagonal system}"'] & \vec{z}(t) = \dots \\
\frac{\mathrm{d}}{\mathrm{d}t}\vec{y_\lambda}(t) = A_{y_\lambda}\vec{y_\lambda}(t) \arrow[r, "\text{Solve a diagonal system}"'] & \vec{y_\lambda}(t) = \dots \arrow[u, "\text{Undo change of variables}"']
\end{tikzcd}$$

Figure 2: A Strategy to Solve for $\vec{z}(t)$

Solution:

$$\vec{z} = \vec{v}_{\lambda_1}y_{\lambda_1} + \vec{v}_{\lambda_2}y_{\lambda_2} \tag{34}$$

$$\vec{z} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} y_{\lambda_1} \\ y_{\lambda_2} \end{bmatrix} \tag{35}$$

We can define the change-of-coordinates matrix from the eigenbasis to our original basis as:

$$V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \tag{36}$$

Changing coordinates to the eigenbasis:

$$\begin{bmatrix} y_{\lambda_1} \\ y_{\lambda_2} \end{bmatrix} = V^{-1}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{37}$$

$$A_{y_\lambda} = V^{-1}A_zV = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 9 & -4 \\ -18 & -6 \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \tag{40}$$

That is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{bmatrix} \tag{41}$$

(e) **Solve the differential equation for $y_{\lambda_i}(t)$ in the eigenbasis.** *Don't forget about the initial conditions!*

Solution: First we get the initial condition:

$$\vec{y_\lambda}(0) = V^{-1}\vec{z}(0) = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{42}$$

Then we solve based on the form of the problem and our previous differential equation experience:

$$\vec{y_\lambda}(t) = \begin{bmatrix} K_1\mathrm{e}^{-9t} \\ K_2\mathrm{e}^{-2t} \end{bmatrix} \tag{43}$$

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Plugging in for the initial condition gives:

$$\vec{y}_\lambda(t) = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} \tag{44}$$

(f) **Convert your solution back into the original coordinates to find $z_i(t)$.**

Solution:

$$\vec{z}(t) = V\vec{y}_\lambda(t) = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix} \tag{45}$$

(g) **(PRACTICE)** In part 1.b of the discussion, we make a simplifying assumption V_{in} transitions from 7 V to 0 V at $t = 0$. We now consider the setting, where the voltage V_{in} transitions from 0 V to 7 V at $t = 0$, i.e we have $V_{\text{in}}(t) = 7$ V for $t \geq 0$.

Find the solution for $z_i(t)$ under the assumption that $V_{\text{in}}(t) = 7$ V for $t \geq 0$ (that is, our system is now nonhomogeneous).

Solution: From eq. (12), we have the general form of the differential equation (valid for $t \geq 0$):

$$\frac{d}{dt}\begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{\text{in}}(t) \tag{46}$$

In this subpart, we have $V_{\text{in}}(t) = 7$, with the initial conditions $V_{C_1}(0) = 0$ and $V_{C_2}(0) = 0$. This system of coupled differential equations in terms of $z_i(t)$ becomes:

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} \tag{47}$$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{48}$$

Concisely, we rewrite this in matrix-vector form as :

$$\frac{d\vec{z}}{dt} = A_z\vec{z} + \vec{b}_z \tag{49}$$

Performing a change of basis as in item **1**.d, we use $\vec{z} = V\vec{y}$, such that :

$$\frac{d}{dt}V\vec{y} = A_zV\vec{y} + \vec{b}_z \tag{50}$$

$$V\frac{d\vec{y}}{dt} = A_zV\vec{y} + \vec{b}_z \tag{51}$$

$$\frac{d\vec{y}}{dt} = V^{-1}A_zV\vec{y} + V^{-1}\vec{b}_z \tag{52}$$

where $V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$. Simplifying, we have:

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} \frac{-3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 21 \\ 0 \end{bmatrix} \tag{53}$$

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$$= \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} -9 \\ 6 \end{bmatrix} \tag{54}$$

This is now a system of uncoupled non-homogeneous differential equations,

$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = -9y_1(t) - 9 \tag{55}$$

$$\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} = -2y_2(t) + 6 \tag{56}$$

with initial conditions

$$\vec{y}(0) = V^{-1}\vec{z}(0) \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{57}$$

With change of variables, we solve eq. (56) to recover the solution:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-9t} - 1 \\ -3\mathrm{e}^{-2t} + 3 \end{bmatrix} \tag{58}$$

Finally, substituting back into the original coordinates, $\vec{z}_t = V\vec{y}(t)$, so:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-9t} - 1 \\ -3\mathrm{e}^{-2t} + 3 \end{bmatrix} \tag{59}$$

$$= \begin{bmatrix} 7 - \mathrm{e}^{-9t} - 6\mathrm{e}^{-2t} \\ 7 + 2\mathrm{e}^{-9t} - 9\mathrm{e}^{-2t} \end{bmatrix} \tag{60}$$

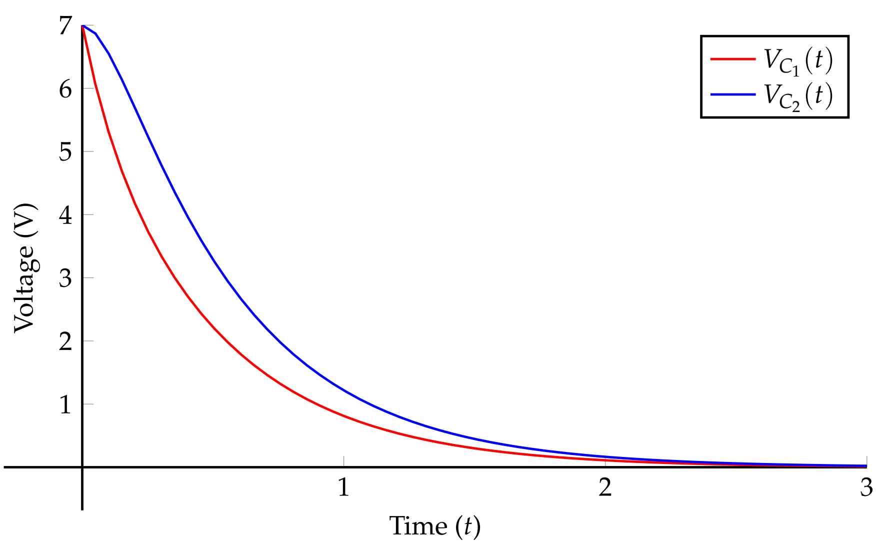

Figure 3: Initial Conditions: $V_{C_1}(0) = 7\,\mathrm{V}$ and $V_{C_2}(0) = 7\,\mathrm{V}$. Homogeneous Case Solution.

Contributors:
- Anant Sahai.
- Regina Eckert.
- Nathan Lambert.
- Pavan Bhargava.
- Kareem Ahmad.
- Neelesh Ramachandran.
- Kumar Krishna Agrawal.

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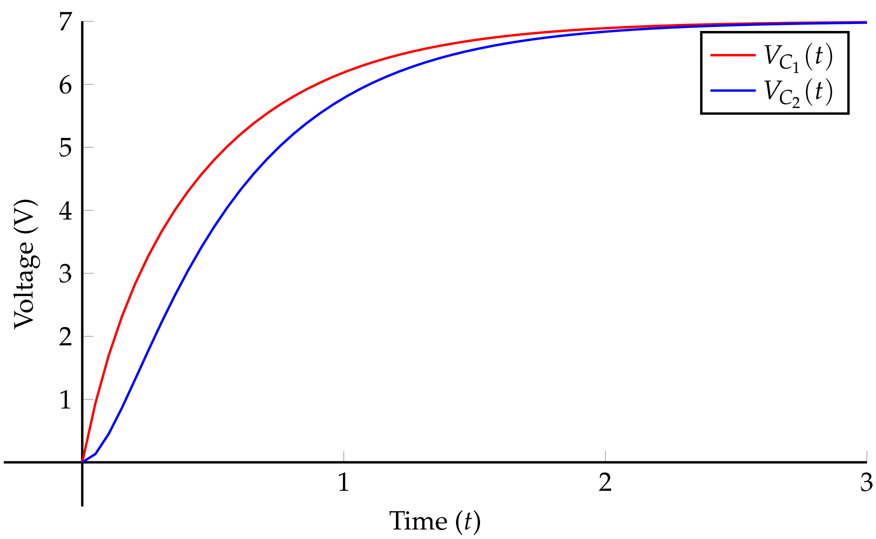

Figure 4: Initial Conditions: $V_{C_1}(0) = 0\,\text{V}$ and $V_{C_2}(0) = 0\,\text{V}$. Non-homogeneous Case Solution.

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