EECS 16B Designing Information Systems and Devices II UC Berkeley Fall 2022

Discussion 7B

The following notes are useful for this discussion: Note 10, Note 11

1. System Identification by Means of Least Squares

(a) Consider the scalar discrete-time system

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at timestep i is $x[i]$, the input applied at timestep i is $u[i]$ and $w[i]$ represents some (small) external disturbance that also participated at timestep i (which we cannot predict or control, it's a purely random disturbance).

Assume that you have measurements for the states $x[i]$ from $i = 0$ to ℓ and also measurements for the controls $u[i]$ from $i = 0$ to $\ell - 1$. Further assume $\ell \geq 2$.

Show that we can set up a linear system as in eq. (2) to find constants a and b. How do we solve this system?

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \tag{2}$$

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1 u_1[i] + b_2 u_2[i] + w[i] \tag{3}$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2.

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

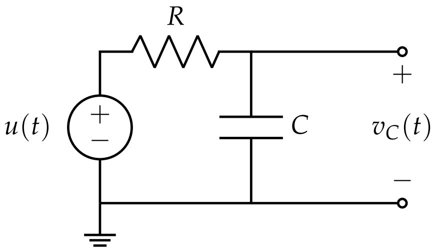
(d) Now consider the two dimensional state case with a single input.

$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{4}$$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$? Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts). *Hint: What work/computation can we reuse across the two problems?*

2. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\,\Omega$, $C = 0.5\,\text{F}$, and $u(t)$ is some function bounded between $-K$ and K for some constant $K \in \mathbb{R}$ (for example $K\cos(t)$). Furthermore assume that $v_C(0) = 0\,\text{V}$ (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = -2v_C(t) + 2u(t) \tag{5}$$

Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit. *HINT: You may want to use the triangle inequality, i.e. $|a+b| \le |a| + |b|$, and the triangle inequality for integrals, i.e. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$. When we use $|\cdot|$ notation here, we will take this to mean the magnitude, rather than the absolute value (since we can be dealing with complex numbers).*

(b) **(PRACTICE)** Now, suppose that in the circuit of part 2.a we replaced the resistor with an inductor as in fig. 1.

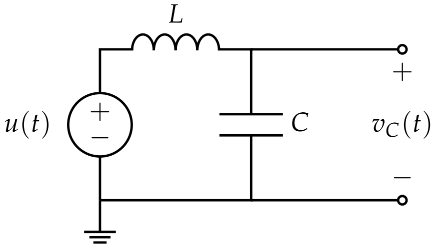


Figure 1: The original circuit with an inductor in place of the resistor.

Let $L = 1\,\text{mH}$. **Repeat part 2.a for the new circuit (with an inductor). Consider the following process to arrive at the result:**

i. **Derive the system of differential equations using KCL, KVL, and NVA. Show that the system is** $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$ **with the initial condition being** $\begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} = \vec{0}$**.**

ii. **Solve the matrix differential equation, using diagonalization if needed. Show that the diagonalized system has a solution**

$$\vec{y}(t) = \begin{bmatrix} \frac{1}{2LC} e^{j\frac{1}{\sqrt{LC}}t} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} u(\theta)\,\mathrm{d}\theta \\ \frac{1}{2LC} e^{-j\frac{1}{\sqrt{LC}}t} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} u(\theta)\,\mathrm{d}\theta \end{bmatrix} \tag{6}$$

where $\vec{y}(t) = V^{-1}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ **for change of basis matrix** V**.** You may use the fact that the eigenvalue, eigenvector pairs of $\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$ are $\left(j\frac{1}{\sqrt{LC}}, \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}\right)$ and $\left(-j\frac{1}{\sqrt{LC}}, \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}\right)$.

iii. **Apply a similar process from part 2.a to show that, if we have a bounded input** $u(t)$**, then the system can grow unboundedly.** When showing that a system is unstable, it suffices to choose a bounded $u(t)$ that makes the system unbounded. We can choose $u(t) = 2\cos\left(\frac{1}{\sqrt{LC}}\right) = e^{j\frac{1}{\sqrt{LC}}t} + e^{-j\frac{1}{\sqrt{LC}}t}$ [1]. *HINT: You may use the fact that* $i_L(t) = y_1(t) + y_2(t)$.

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ *and* $i_L(t)$ *as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.*

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[1] The natural frequency of this system is $\omega_n = \frac{1}{\sqrt{LC}}$. If we excite this system at a period equal to the natural frequency, we can make it grow unboundedly. This is similar to pushing a swing at the same rate it swings, which makes it swing farther.

(c) Thus far, we have dealt with continuous systems so it also makes sense to consider discrete systems. Consider the discrete system

$$x[i+1] = 2x[i] + u[i] \tag{7}$$

with $x[0] = 0$.

Is the system stable or unstable? If unstable, find a bounded input sequence $u[i]$ that causes the system to "blow up".

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kareem Ahmad.
- Sidney Buchbinder.