Midterm Review

1. Hambley P3.70

For the circuit in fig. 1, **determine $i(t)$, $v_L(t)$, $v(t)$, the energy stored in the capacitance, the energy stored in the inductance, and the total stored energy**, given that $v_C(t) = 40\cos(1000t)$ V. (The argument of the cosine function is in radians.)

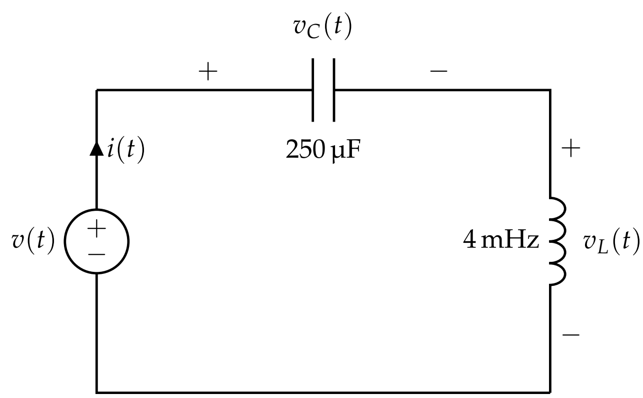


Figure 1

Show that the total stored energy is constant with time. Comment on the results.

Solution: This problem is pretty mechanical and will rely on the the element equations for capacitors and inductors.

Solving for $i(t)$:

$$i(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \tag{1}$$
$$= C\frac{\mathrm{d}}{\mathrm{d}t}(40\cos(1000t)) \tag{2}$$
$$= (250 \times 10^{-6})(-40000\sin 1000t) \tag{3}$$
$$= -10\sin(1000t) \tag{4}$$

Solving for $v_L(t)$:

$$v_L(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \tag{5}$$
$$= L\frac{\mathrm{d}}{\mathrm{d}t}(-10\sin(1000t)) \tag{6}$$
$$= (0.0004(-10000\cos 1000t)) \tag{7}$$
$$= -40\cos(1000t) \tag{8}$$

Solving for $v(t)$:

$$v(t) = v_C(t) + v_L(t) \tag{9}$$

$$= 40\cos(1000t) - 40\cos(1000t) \tag{10}$$
$$= 0 \tag{11}$$

Solving for $U_C(t)$:

$$U_C(t) = \frac{1}{2}Cv_C(t)^2 \tag{12}$$
$$= \frac{1}{2}(250\times 10^{-6})(40\cos(1000t))^2 \tag{13}$$
$$= 0.2\cos^2(1000t) \tag{14}$$

Solving for $U_L(t)$:

$$U_L(t) = \frac{1}{2}Li(t)^2 \tag{15}$$
$$= \frac{1}{2}(0.004)(-10\sin(1000t))^2 \tag{16}$$
$$= 0.2\sin^2(1000t) \tag{17}$$

Solving for $U(t)$:

$$U(t) = U_C(t) + U_L(t) \tag{18}$$
$$= 0.2\cos^2(1000t) + 0.2\sin^2(1000t) \tag{19}$$
$$= 0.2(\cos^2(1000t) + \sin^2(1000t)) \tag{20}$$
$$= 0.2(1) \tag{21}$$
$$= 0.2\,\mathrm{J} \tag{22}$$

Notice that the total power stored between the capacitor and inductor is constant at any given time, and the power will be transferred back and forth between the capacitor and inductor.

2. Hambley P4.34

Consider the circuit shown in fig. 2. The initial current for $i_L(0_-) = 0$.

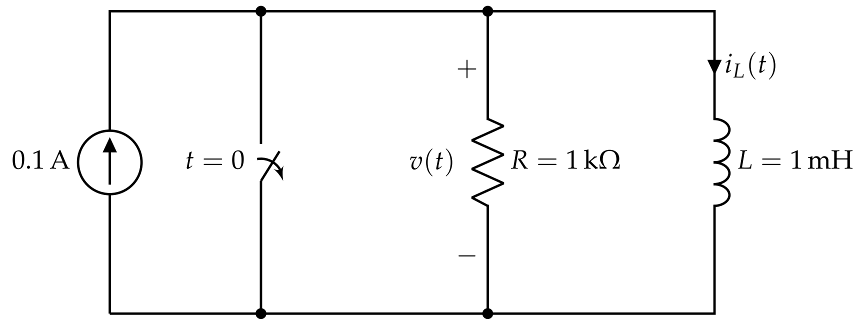


Figure 2: RL Circuit

Find expressions for $i_L(t)$ and $v(t)$ for $t \geq 0$ and qualitatively sketch to scale versus time.

Solution: The initial condition is $i_L(0+) = 0$ because we are given that $i_L(0-) = 0$ and we also know that the current across an inductor cannot change instantaneously. When the switch is closed all the current will flow through the switch branch since it is a short circuit. When it opens at $t = 0$, the current will begin to flow through the other branches. Writing out KCL, we have:

$$I_s = i_R(t) + i_L(t) \tag{23}$$

Plugging in Ohm's law and using the fact that $v(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}$ we get:

$$I_s = \frac{v(t)}{R} + i_L(t) \tag{24}$$

$$I_s = \frac{L}{R}\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + i_L(t) \tag{25}$$

$$\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + \frac{R}{L}i_L(t) = \frac{R}{L}I_s \tag{26}$$

We will first solve for the homogeneous/complementary solution:

$$\frac{\mathrm{d}i_\mathrm{h}(t)}{\mathrm{d}t} + \frac{R}{L}i_\mathrm{h}(t) = 0 \tag{27}$$

$$\frac{\mathrm{d}i_\mathrm{h}(t)}{\mathrm{d}t} = -\frac{R}{L}i_\mathrm{h}(t) \tag{28}$$

$$i_\mathrm{h}(t) = A\mathrm{e}^{-\frac{R}{L}t} \tag{29}$$

The particular solution can be solved for by analyzing the DC steady state of our circuit since our current source is DC. Recall in DC steady state:

- Capacitors are treated as open circuits
- Inductors are treated as short circuits

Thus, we get that the inductor will short and thus all the current 0.1 A will flow through that branch and no current will flow through the resistor. Therefore, $i_\mathrm{p} = 0.1$. Since $i_L(t) = i_\mathrm{h}(t) + i_\mathrm{p}(t)$, we have:

$$i_L(t) = 0.1 + A\mathrm{e}^{-\frac{R}{L}t} \tag{30}$$

Now using the initial condition we see that:

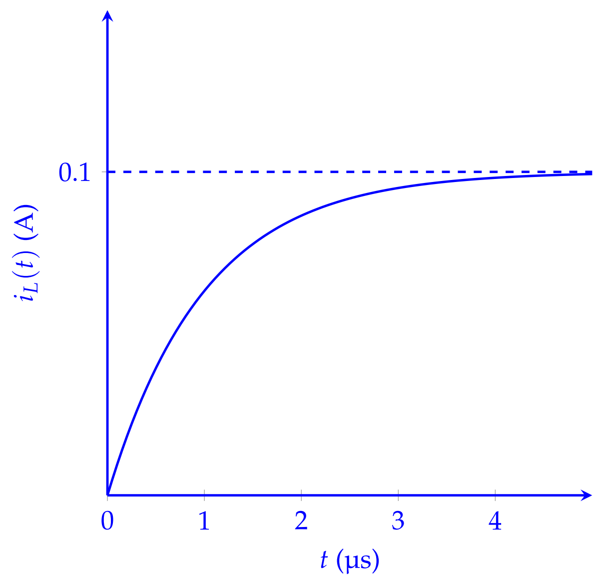
$$i_L(0) = 0.1 + A\mathrm{e}^{0\frac{R}{L}t} \tag{31}$$
$$0 = 0.1 + A \tag{32}$$
$$A = -0.1 \tag{33}$$

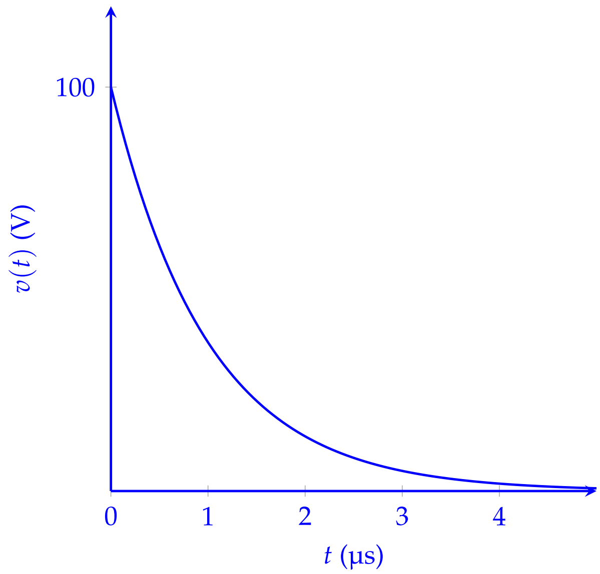
Putting this together, we get:

$$i_L(t) = 0.1(1 - \mathrm{e}^{-10^6 t}) \tag{34}$$

Solving for $v(t)$:

$$v(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} \tag{35}$$
$$v(t) = L\frac{\mathrm{d}}{\mathrm{d}t} \tag{36}$$
$$v(t) = L \times 10^6 \mathrm{e}^{-10^6 t} \tag{37}$$
$$v(t) = 100\mathrm{e}^{-10^6 t} \tag{38}$$




100
$v(t)$ (V)
0
1
2
3
4
t (µs)

3. Hambley P5.85

Suppose you are given the following two terminal circuit in fig. 3.

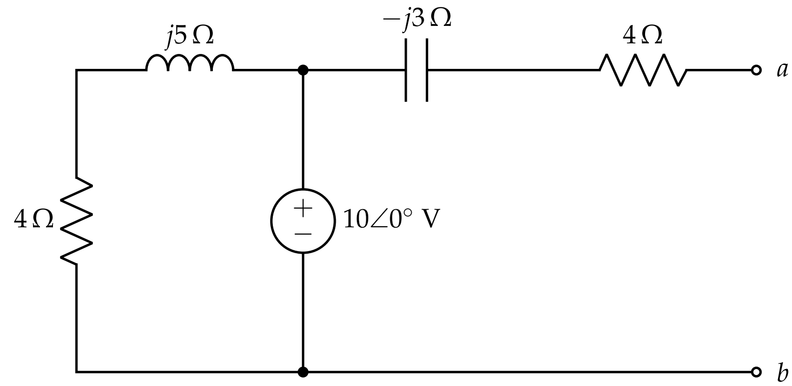


Figure 3: Two Terminal Circuit

Find the Thevenin voltage, Thevenin impedance, and Norton current for the cirucit.

Solution: First notice that in this configuration, no current will flow thorugh the $-j3\,\Omega$ capacitor or the $4\,\Omega$ resistor in series with it so there is no voltage drop across either element. This gives us that the voltage between nodes a and b is $V_{\text{TH}} = 10\angle 0°$.

The Norton current is solved for by shorting nodes a and b and solving for the current of this short.

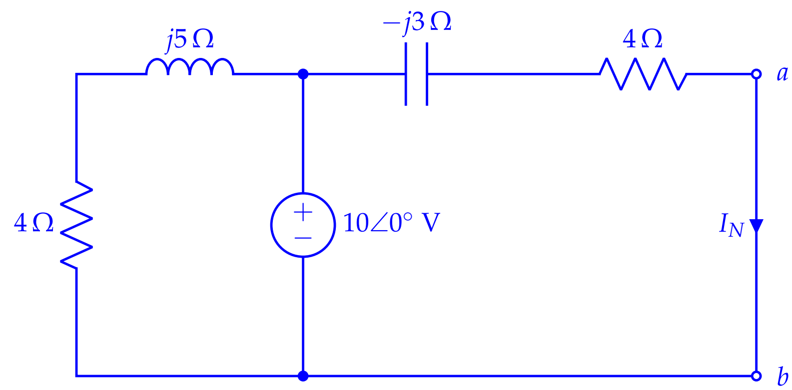


Figure 4: Shorted Two Terminal Circuit

$$I_N = \frac{10\angle 0°}{-\text{j}3 + 4} \tag{39}$$
$$= \frac{10\angle 0°}{5\angle -37°} \tag{40}$$
$$= 2\angle 37° \tag{41}$$

The Thevenin impedance can be calculated as follows:

$$Z_{\text{TH}} = \frac{V_{\text{TH}}}{I_{\text{N}}} \tag{42}$$

$$= \frac{10\angle 0^\circ}{2\angle 37^\circ} \tag{43}$$

$$= 5\angle -37^\circ \tag{44}$$

Equivalent Circuits can be drawn as follows:

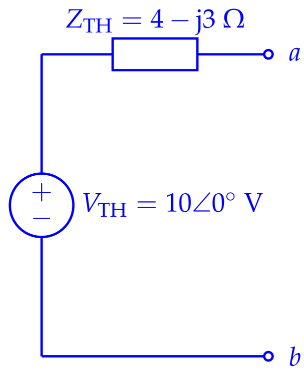


Figure 5: Thevenin Equivalent

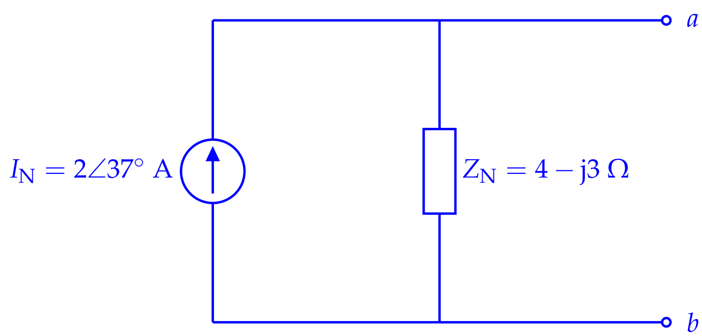


Figure 6: Norton Equivalent

4. Hambley P6.57

The circuit shown fig. 7 has $R_1 = R_2 = 2\,\mathrm{k}\Omega$ and $C = \frac{1}{\pi}\,\mu\mathrm{F}$.

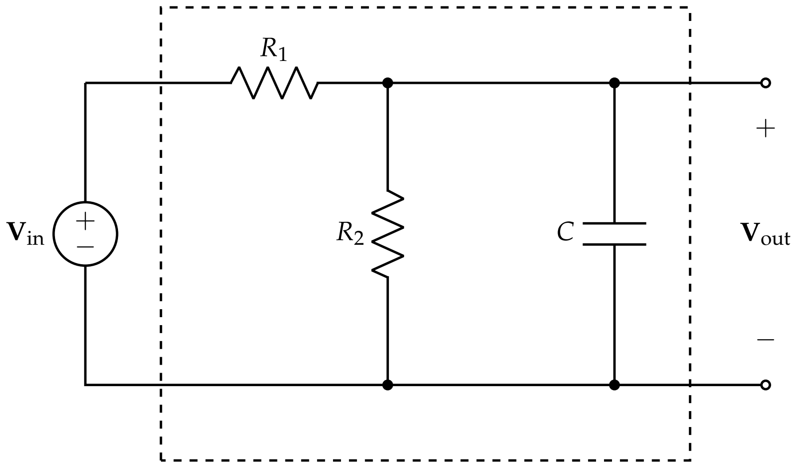


Figure 7

Solve for the transfer function $H(f) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}$, calculate the half-power frequency, and analyze the magnitude and phase of $H(f)$ as $f \to 0$ and $f \to \infty$.

Solution: Recall, we first need to convert our circuit into our phasor domain with impedances. Then we will observe that the resistor R_2 and capacitor are in parallel so we can simplify into an equivalent impedance and then apply the voltage divider to get V_{out}.

$$H(f) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} \tag{45}$$

$$= \frac{(R_2||C)}{R_1 + (R_2||C)} \tag{46}$$

$$= \frac{(R_2\left(\frac{1}{\mathrm{j}2\pi fC}\right))}{R_2 + \left(\frac{1}{\mathrm{j}2\pi fC}\right)} \frac{1}{R_1 + \frac{(R_2\left(\frac{1}{\mathrm{j}2\pi fC}\right))}{R_2+\left(\frac{1}{\mathrm{j}2\pi fC}\right)}} \tag{47}$$

$$= \left(\frac{R_2}{1 + \mathrm{j}2\pi fR_2C}\right)\left(\frac{1}{R_1 + \frac{R_2}{1+\mathrm{j}2\pi fR_2C}}\right) \tag{48}$$

$$= \frac{R_2}{(R_1 + R_2) + \mathrm{j}2\pi fR_1R_2C} \tag{49}$$

$$= \frac{\frac{R_2}{R_1+R_2}}{1 + \mathrm{j}f\frac{2\pi R_1R_2C}{R_1+R_2}} \tag{50}$$

Notice that this transfer is in the same form as a low-pass filter where the maximum gain is $\frac{R_2}{R_1+R_2} = \frac{1}{2}$.

The half power frequency can be pattern matched to $\frac{R_1+R_2}{2\pi R_1R_2C}$, but we can also directly solve for it by calculating the frequency at which the magnitude of the transfer function is $\frac{1}{\sqrt{2}}$ the maximum magnitude:

$$|H(f_c)| = \frac{1}{\sqrt{2}}\frac{1}{2} \tag{51}$$

$$\frac{1}{2}\left|\frac{1}{1+\mathrm{j}f_c\frac{2\pi R_1 R_2 C}{R_1+R_2}}\right| = \frac{1}{\sqrt{2}}\frac{1}{2} \tag{52}$$

$$\left|\frac{1}{1+\mathrm{j}f_c\frac{2\pi R_1 R_2 C}{R_1+R_2}}\right| = \frac{1}{\sqrt{2}} \tag{53}$$

In order the equation above to hold true we need $f_c\frac{2\pi R_1 R_2 C}{R_1+R_2} = \pm 1$. Since all our values are positive, we want it to be equal to 1. In this case, that leaves us with:

$$f_c = \frac{R_1+R_2}{2\pi R_1 R_2 C} \tag{54}$$

This is the same result that we got from pattern matching from the low-pass filter. Plugging in values we get $f_c = 500$ Hz.

If we take $f \to 0$ we see that the transfer function becomes:

$$H(f) = \frac{R_2}{R_1+R_2} \tag{55}$$

The magnitude as a result is $\frac{R_2}{R_1+R_2} = \frac{1}{2}$ and the phase is $0°$.

If we take $f \to \infty$ we see that the transfer function becomes:

$$H(f) = \frac{\frac{R_2}{R_1+R_2}}{1+\mathrm{j}\infty} \tag{56}$$

The magnitude of this would be $\frac{1}{2}\frac{1}{\sqrt{1^2+\infty^2}} = 0$ and the phase would $\angle\left(\frac{1}{2}\right) - \angle(1+\mathrm{j}\infty) = 0° - 90° = -90°$.

This aligns with the fact that we have a low-pass filter.

5. Hambley P6.82

Consider the parallel resonant circuit shown in fig. 8.

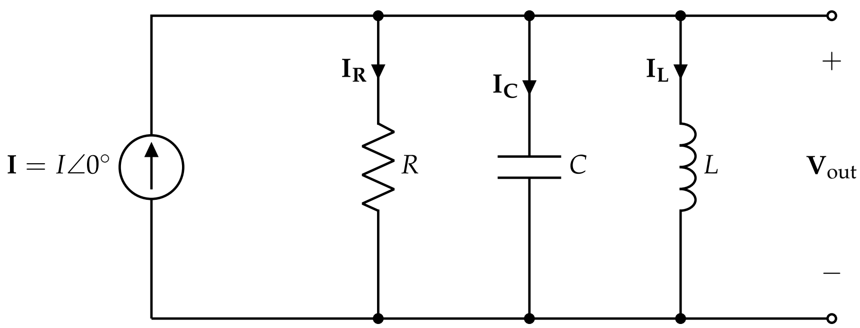


Figure 8: Parallel Resonant Circuit

Determine the L and C values, given $R = 2\,\text{k}\Omega$, $f_0 = 8\,\text{MHz}$, and $B = 500\,\text{kHz}$. Then draw a phasor diagram showing the currents through each of the elements in the circuit at resonance given that $\mathbf{I} = 10^{-3}\angle 0°$.

Solution: First we will solve for the quality factor given by:

$$Q_p = \frac{f_0}{B} = \frac{8 \times 10^6}{500 \times 10^3} = 16 \tag{57}$$

Using the different representations of the quality factor for the parallel resonant circuit, we can solve for the values of L and C:

$$Q_p = \frac{R}{2\pi f_0 L} \tag{58}$$

$$L = \frac{R}{2\pi f_0 Q_p} \tag{59}$$

$$L = \frac{2000}{2\pi(8 \times 10^6)(16)} \tag{60}$$

$$L = 2.49 \times 10^{-6}\ \text{H} \tag{61}$$

$$L = 2.49\,\mu\text{H} \tag{62}$$

Similarly, we can solve for C:

$$Q_p = 2\pi f_0 RC \tag{63}$$

$$C = \frac{Q_p}{2\pi f_0 R} \tag{64}$$

$$C = \frac{16}{2\pi(8 \times 10^6)(3000)} \tag{65}$$

$$C = 1.59 \times 10^{-10}\ \text{F} \tag{66}$$

$$C = 159\,\text{pF} \tag{67}$$

One way to approach this problem is to first solve for $\mathbf{V}_{\text{out}}$ and then use that to calculate the current phasor through element.

$$\mathbf{V}_{\text{out}} = IZ_p = (Z_R || Z_C || Z_L) \tag{68}$$

$$\mathbf{V}_{\text{out}} = I\frac{R}{1 - \mathrm{j}Q_p\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} \tag{69}$$

Since our circuit is in resonance, we get:

$$\mathbf{V}_{\text{out}} = IR = (10^{-3}\angle 0^\circ)(2000) = 2\angle 0^\circ \text{ V} \tag{70}$$

Now, we can solve for the element currents:

$$\mathbf{I_R} = \frac{\mathbf{V}_{\text{out}}}{R} = \frac{2\angle 0^\circ}{2000} = 10^{-3}\angle 0^\circ \text{ A} \tag{71}$$

$$\mathbf{I_L} = \frac{\mathbf{V}_{\text{out}}}{\mathrm{j}2\pi f_0 L} = \frac{2\angle 0^\circ}{2\pi(8\times 10^6)2.49\times 10^{-6}\angle 90^\circ} = 0.016\angle -90^\circ \text{ A} \tag{72}$$

$$\mathbf{I_C} = \frac{\mathbf{V}_{\text{out}}}{\frac{1}{\mathrm{j}2\pi f_0 C}} = (2\angle 0^\circ)(2\pi(8\times 10^6)(159\times 10^{-12})\angle 90^\circ) = 0.016\angle 90^\circ \text{ A} \tag{73}$$