Discussion 8B

The following notes are useful for this discussion: Note 10 and Note 11.

1. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i+1] = 0.9x[i] + u[i] + w[i] \tag{1}$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i.

Is the system stable? If $|w[i]| \leq \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

(b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. **Given a specific λ, you want the system to behave like:**

$$x[i+1] = \lambda x[i] + w[i]? \tag{2}$$

To do so, how would you pick f?

NOTE: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.

(c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**

(d) **What if instead of a 0.9, we had a 3 in the original eq. (1). Would system stability change? Would our ability to control λ change?**

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{3}$$

where we further assume that B is an invertible square matrix. Futher, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.

Given a specific A_{CL} we want the system to behave like:

$$\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i] + \vec{w}[i]? \tag{4}$$

How would you pick F given knowledge of A, B and the desired goal dynamics A_{CL}? Will this work for any desired A_{CL}?

2. Controlling states by designing sequences of inputs

Consider the following matrix, with a simple structure (what does it do when it acts on a vector?):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]. \tag{6}$$

(a) **Show that this system is controllable.**

(b) Suppose that we would like to "place" the eigenvalues of A_{CL} (the closed loop A matrix) to be at $0.1, 0.2, 0.3, 0.4$. That is, we would like to implement a feedback control law such that the eigenvalues of A_{CL} will be $0.1, 0.2, 0.3, 0.4$, where our new system would be given by

$$\vec{x}[i+1] = A_{CL}\vec{x}[i] \tag{7}$$

We define the characteristic polynomial of a matrix M to be

$$p_M(\lambda) = \det\{\lambda I - M\} \tag{8}$$

If we were to place the eigenvalues of A_{CL} at $0.1, 0.2, 0.3, 0.4$, what will $p_{A_{CL}}(\lambda)$ be?

(c) Since A is in controllable canonical form (CCF), we know that the characteristic polynomial of the matrix will be

$$p_A(\lambda) = \det\{\lambda I - A\} = \lambda^4 - a_4\lambda^3 - a_3\lambda^2 - a_2\lambda - a_1 \tag{9}$$

Given a feedback control law

$$\vec{u}[i] = \underbrace{\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}}_{F} \vec{x}[i] \tag{10}$$

determine the values of f_1, f_2, f_3, f_4 in terms of a_1, a_2, a_3, a_4 so that the eigenvalues of A_{CL} will be $0.1, 0.2, 0.3, 0.4$.

Contributors:

- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kumar Krishna Agrawal.
- Anish Muthali.

