Discussion 8B

The following notes are useful for this discussion: Note 10 and Note 11.

1. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i+1] = 0.9x[i] + u[i] + w[i] \tag{1}$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i.

Is the system stable? If $|w[i]| \le \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

Solution: The system is stable, as $\lambda = 0.9 \implies |\lambda| < 1$. We can say that $|x[i]|$ is bounded at all time if the disturbance is bounded. Unrolling the system's recursion and extrapolating the general form,

$$x[0] = 0 \tag{2}$$
$$x[1] = w[0] \tag{3}$$
$$x[2] = 0.9w[0] + w[1] \tag{4}$$
$$x[3] = 0.9^2 w[0] + 0.9w[1] + w[2] \tag{5}$$
$$\vdots \tag{6}$$
$$x[i] = \sum_{k=0}^{i-1} 0.9^k w[i-k-1]. \tag{7}$$

We can check that this form works by plugging it into our recursion:

$$x[i+1] = 0.9x[i] + w[i] = 0.9\left(\sum_{k=0}^{i-1} 0.9^k w[i-k-1]\right) + w[i] \tag{8}$$
$$= \sum_{k=0}^{i-1} 0.9^{k+1} w[i-k-1] + w[i] = \sum_{k=0}^{i} 0.9^k w[i-k] \tag{9}$$

which is exactly what our formula predicts. So,

$$|x[i]| = \left|\sum_{k=0}^{i-1} 0.9^k w[i-k-1]\right| \le \sum_{k=0}^{i-1} \left|0.9^k w[i-k-1]\right| = \sum_{k=0}^{i-1} 0.9^k \epsilon. \tag{10}$$

In the limit as $i \to \infty$, by the geometric series formula,

$$|x[i]| \le \frac{\epsilon}{1-0.9} = 10\epsilon \tag{11}$$

(b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. **Given a specific λ, you want the system to behave like:**

$$x[i+1] = \lambda x[i] + w[i]? \tag{12}$$

To do so, how would you pick f?

NOTE: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.

Solution: We can control the system to have any value of λ, as long as we're not limited on the values of f.

$$x[i+1] = 0.9x[i] + fx[i] + w[i] = \lambda x[i] + w[i]. \tag{13}$$

Fitting terms, $f = \lambda - 0.9$. Note we can get a $\lambda > 1$ if we so desire; there is nothing stopping us from putting arbitrarily big/small λ by the choice of f.

(c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**

Solution: From eq. (12), in order to have the minimum bound on $|x[i]|$, $\lambda = 0$. To get this λ, $f = -0.9$. In the limit as $i \to \infty$ in this case,

$$|x[i]| \leq \frac{\epsilon}{1-0} = \epsilon \tag{14}$$

The minimum bound on $|x(i)| = \epsilon$ is the same bound as on the disturbance.

(d) **What if instead of a 0.9, we had a 3 in the original eq. (1). Would system stability change? Would our ability to control λ change?**

Solution: If our system were now,

$$x[i+1] = 3x[i] + u[i] + w[i], \tag{15}$$

the system would no longer be stable. However, we can still choose any λ using closed loop feedback. In this case, $f = \lambda - 3$.

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{16}$$

where we further assume that B is an invertible square matrix. Futher, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.

Given a specific A_{CL} we want the system to behave like:

$$\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i] + \vec{w}[i]? \tag{17}$$

How would you pick F given knowledge of A, B and the desired goal dynamics A_{CL}? Will this work for any desired A_{CL}?

Solution: Since in this case our input is the same rank as our output, we can arbitrarily choose the matrix A_{CL}. As long as B is invertible (as given), we can define:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{18}$$

$$= A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i] \quad (19)$$
$$= (A + BF)\vec{x}[i] + \vec{w}[i] \quad (20)$$
$$= A_{\text{CL}}\vec{x}[i] + \vec{w}[i] \quad (21)$$

Therefore, matching terms,

$$A + BF = A_{\text{CL}} \implies F = B^{-1}(A_{\text{CL}} - A). \quad (22)$$

2. Controlling states by designing sequences of inputs

Consider the following matrix, with a simple structure (what does it do when it acts on a vector?):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{23}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]. \tag{24}$$

(a) **Show that this system is controllable.**

Solution: We have that $\mathcal{C} = \begin{bmatrix} \vec{b} & A\vec{b} & A^2\vec{b} & A^3\vec{b} \end{bmatrix}$ where

$$\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{25}$$

$$A\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ a_4 \end{bmatrix} \tag{26}$$

$$A^2\vec{b} = \begin{bmatrix} 0 \\ 1 \\ a_4 \\ a_3 + a_4^2 \end{bmatrix} \tag{27}$$

$$A^3\vec{b} = \begin{bmatrix} 1 \\ a_4 \\ a_3 + a_4^2 \\ a_2 + 2a_3a_4 + a_4^3 \end{bmatrix} \tag{28}$$

Since each of these vectors have one less nonzero entry than the one above it, the vectors are all linearly independent and $\mathcal{C}$ is full rank. Hence, the system is controllable.

(b) Suppose that we would like to "place" the eigenvalues of A_{CL} (the closed loop A matrix) to be at $0.1, 0.2, 0.3, 0.4$. That is, we would like to implement a feedback control law such that the eigenvalues of A_{CL} will be $0.1, 0.2, 0.3, 0.4$, where our new system would be given by

$$\vec{x}[i+1] = A_{CL}\vec{x}[i] \tag{29}$$

We define the characteristic polynomial of a matrix M to be

$$p_M(\lambda) = \det\{\lambda I - M\} \tag{30}$$

If we were to place the eigenvalues of A_{CL} at $0.1, 0.2, 0.3, 0.4$, what will $p_{A_{CL}}(\lambda)$ be?

Solution: We know that 0.1, 0.2, 0.3, 0.4 will be roots of the polynomial $\det\{\lambda I - A_{CL}\} = p_{A_{CL}}(\lambda)$. Hence, it is the case that we can write

$$p_{A_{CL}}(\lambda) = (\lambda - 0.1)(\lambda - 0.2)(\lambda - 0.3)(\lambda - 0.4) \tag{31}$$
$$= \lambda^4 - \lambda^3 + 0.35\lambda^2 - 0.05\lambda + 0.0024 \tag{32}$$

(c) Since A is in controllable canonical form (CCF), we know that the characteristic polynomial of the matrix will be

$$p_A(\lambda) = \det\{\lambda I - A\} = \lambda^4 - a_4\lambda^3 - a_3\lambda^2 - a_2\lambda - a_1 \tag{33}$$

Given a feedback control law

$$\vec{u}[i] = \underbrace{\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}}_{F} \vec{x}[i] \tag{34}$$

determine the values of f_1, f_2, f_3, f_4 in terms of a_1, a_2, a_3, a_4 so that the eigenvalues of A_{CL} will be 0.1, 0.2, 0.3, 0.4.

Solution: When we apply the feedback control law, our closed loop matrix will be

$$A_{CL} = A + \vec{b}F \tag{35}$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f_1 & f_2 & f_3 & f_4 \end{bmatrix} \tag{36}$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 + f_1 & a_2 + f_2 & a_3 + f_3 & a_4 + f_4 \end{bmatrix} \tag{37}$$

Here, the characteristic polynomial will be

$$p_{A_{CL}}(\lambda) = \lambda^4 - (a_4 + f_4)\lambda^3 - (a_3 + f_3)\lambda^2 - (a_2 + f_2)\lambda - (a_1 + f_1) \tag{38}$$

However, we want our characteristic polynomial to be as in eq. (32). To achieve this, we can pattern match coefficients of matching order in eq. (32) and eq. (38) to obtain

$$-(a_4 + f_4) = -1 \tag{39}$$
$$-(a_3 + f_3) = 0.35 \tag{40}$$
$$-(a_2 + f_2) = -0.05 \tag{41}$$
$$-(a_1 + f_1) = 0.0024 \tag{42}$$

Solving for f_1, f_2, f_3, f_4, we have

$$f_4 = 1 - a_4 \tag{43}$$
$$f_3 = -0.35 - a_3 \tag{44}$$

$$f_2 = 0.05 - a_2 \tag{45}$$
$$f_1 = -0.0024 - a_1 \tag{46}$$

so the feedback control law is

$$\vec{u}[i] = \begin{bmatrix} 1 - a_4 & -0.35 - a_3 & 0.05 - a_2 & -0.0024 - a_1 \end{bmatrix} \vec{x}[i] \tag{47}$$

Contributors:

- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kumar Krishna Agrawal.
- Anish Muthali.