Discussion 11A

The following note is useful for this discussion: Note 16

1. Computing the SVD: A "Tall" Matrix Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}. \tag{1}$$

Here, we expect $U \in \mathbb{R}^{3\times 3}$, $\Sigma \in \mathbb{R}^{3\times 2}$, and $V \in \mathbb{R}^{2\times 2}$ (recall that U and V must be square since they are orthonormal matrices).

In this problem, we will walk through the SVD algorithm, prove some important theorems about the SVD matrices and column/null spaces, and consider an alternate way to approach the SVD.

(a) Let's start by trying to write A as an outer product in the form of $\sigma\vec{u}\vec{v}^{\top}$ where both $\vec{u}$ and $\vec{v}^{\top}$ have unit norm. *(HINT: Are the columns of A linearly independent or dependent? What does that tell us about how we can represent them?)*

(b) In this part, we will walk through Algorithm 7 in Note 16. This algorithm applies for a general matrix $A \in \mathbb{R}^{m\times n}$.

i. **Find $r := \operatorname{rank}(A)$. Compute $A^{\top}A$ and diagonalize it using the spectral theorem (i.e. find V and Λ).**

ii. **Unpack $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ and unpack $\Lambda := \begin{bmatrix} \Lambda_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{bmatrix}$.**

iii. **Find $\Sigma_r := \Lambda_r^{1/2}$ and then find $\Sigma := \begin{bmatrix} \Sigma_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$.**

iv. **Find $U_r := AV_r\Sigma_r^{-1}$, where $U_r \in \mathbb{R}^{3\times 1}$ and then extend the basis defined by columns of U_r to find $U \in \mathbb{R}^{3\times 3}$.**
(HINT: How can we extend a basis, and why is that needed here?)

v. **Use the previous parts to write the full SVD of A.**

vi. **If we were to calculate the SVD of our matrix using a calculator, are we *guaranteed* to always get the same SVD? Why or why not?**

(c) We now want to create the SVD of $A^\top$. **What are the relationships between the matrices composing A and the matrices composing $A^\top$?**

© UCB EECS 16B, Fall 2022. All Rights Reserved. This may not be publicly shared without explicit permission.

(d) **Show, for a general matrix** $A \in \mathbb{R}^{m \times n}$ **with** $\operatorname{rank}(A) = r$ **and** $A = U\Sigma V^{\top}$**, that** $\operatorname{Null}(A) = \operatorname{Col}(V_{n-r})$**. Then, find a basis for the null space of** A **in eq. (1).** *(HINT: How do we show two sets are equal? Try and use that approach here. Consider the outer product summation form for the SVD. Also, consider using the rank-nullity theorem that* $\dim \operatorname{Col}(A) + \dim \operatorname{Null}(A) = n$*.)*

(e) **Show, for a general matrix** $A \in \mathbb{R}^{m \times n}$ **with** $\operatorname{rank}(A) = r$ **and** $A = U\Sigma V^{\top}$**, that** $\operatorname{Col}(A) = \operatorname{Col}(U_r)$**. Then, find a basis for the range (or column space) of** A**.**

© UCB EECS 16B, Fall 2022. All Rights Reserved. This may not be publicly shared without explicit permission.

(f) **(PRACTICE) Show, for a general matrix $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$ and $A = U\Sigma V^\top$, that $\operatorname{Null}(A^\top) = \operatorname{Col}(U_{m-r})$ and $\operatorname{Col}(A^\top) = \operatorname{Col}(V_r)$. Then show:**

i. $\dim \operatorname{Col}(A) + \dim \operatorname{Null}(A^\top) = n$**,**

ii. and $\operatorname{Col}(A)$ and $\operatorname{Null}(A^\top)$ are orthogonal.

(g) Suppose A was a wide matrix. Instead of finding $A^\top A$, we may want to find the SVD by computing $AA^\top$. The original Algorithm 7 from Note 16, in its entirety, is shown below:

Algorithm 1 Constructing the SVD

function FULLSVD($A \in \mathbb{R}^{m \times n}$)
- $r := \text{RANK}(A)$
- $(V, \Lambda) := \text{DIAGONALIZE}(A^\top A)$ ▷ Sorted so that $\Lambda_{11} \geq \cdots \geq \Lambda_{nn}$
- Unpack $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$
- Unpack $\Lambda := \begin{bmatrix} \Lambda_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix}$
- $\Sigma_r := \Lambda_r^{1/2}$
- Pack $\Sigma := \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$
- $U_r := AV_r\Sigma_r^{-1}$
- $U := \text{EXTENDBASIS}(U_r, \mathbb{R}^m)$
- **return** (U, Σ, V)

end function

Write a modified version of Algorithm 7 where you solve for the SVD of A using $AA^\top$ instead of $A^\top A$. *(HINT: Consider replacing every instance of "A" in $A^\top A$ with "$A^\top$". What happens? How can we use the result from the* ***1.c*** *part?)*

© UCB EECS 16B, Fall 2022. All Rights Reserved. This may not be publicly shared without explicit permission.

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Druv Pai.
- John Maidens.

© UCB EECS 16B, Fall 2022. All Rights Reserved. This may not be publicly shared without explicit permission.