The following note is useful for this discussion: Note 16

1. Computing the SVD: A "Tall" Matrix Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$
 (1)

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Here, we expect $U \in \mathbb{R}^{3\times 3}$, $\Sigma \in \mathbb{R}^{3\times 2}$, and $V \in \mathbb{R}^{2\times 2}$ (recall that U and V must be square since they are orthonormal matrices).

In this problem, we will walk through the SVD algorithm, prove some important theorems about the SVD matrices and column/null spaces, and consider an alternate way to approach the SVD.

(a) Let's start by trying to write A as an outer product in the form of $\sigma \vec{u} \vec{v}^{\top}$ where both \vec{u} and \vec{v}^{\top} have unit norm. (*HINT: Are the columns of A linearly independent or dependent? What does that tell us about how we can represent them?*)

- (b) In this part, we will walk through Algorithm 7 in Note 16. This algorithm applies for a general matrix $A \in \mathbb{R}^{m \times n}$.
 - i. Find $r := \operatorname{rank}(A)$. Compute $A^{\top}A$ and diagonalize it using the spectral theorem (i.e. find V and Λ).

ii. Unpack
$$V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$$
 and unpack $\Lambda := \begin{bmatrix} \Lambda_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix}$.
iii. Find $\Sigma_r := \Lambda_r^{1/2}$ and then find $\Sigma := \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$.

- iv. Find $U_r := AV_r \Sigma_r^{-1}$, where $U_r \in \mathbb{R}^{3 \times 1}$ and then extend the basis defined by columns of U_r to find $U \in \mathbb{R}^{3 \times 3}$. (*HINT: How can we extend a basis, and why is that needed here?*)
- v. Use the previous parts to write the full SVD of *A*.
- vi. If we were to calculate the SVD of our matrix using a calculator, are we *guaranteed* to always get the same SVD? Why or why not?

(c) We now want to create the SVD of A^{\top} . What are the relationships between the matrices composing A and the matrices composing A^{\top} ?

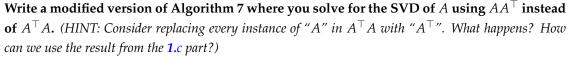
(d) Show, for a general matrix $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$ and $A = U\Sigma V^{\top}$, that $\operatorname{Null}(A) = \operatorname{Col}(V_{n-r})$. Then, find a basis for the null space of A in eq. (1). (*HINT: How do we show two sets are equal? Try and use that approach here. Consider the outer product summation form for the SVD. Also, consider using the rank-nullity theorem that dim \operatorname{Col}(A) + \operatorname{dim Null}(A) = n.)*

(e) Show, for a general matrix $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$ and $A = U\Sigma V^{\top}$, that $\operatorname{Col}(A) = \operatorname{Col}(U_r)$. Then, find a basis for the range (or column space) of A.

- (f) (**PRACTICE**) Show, for a general matrix $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$ and $A = U\Sigma V^{\top}$, that $\operatorname{Null}(A^{\top}) = \operatorname{Col}(U_{m-r})$ and $\operatorname{Col}(A^{\top}) = \operatorname{Col}(V_r)$. Then show:
 - i. dim Col(A) + dim Null(A^{\top}) = n,
 - ii. and $\operatorname{Col}(A)$ and $\operatorname{Null}(A^{\top})$ are orthogonal.

(g) Suppose *A* was a wide matrix. Instead of finding $A^{\top}A$, we may want to find the SVD by computing AA^{\top} . The original Algorithm 7 from Note 16, in its entirety, is shown below:

Algorithm 1 Constructing the SVD		
1: function FULLSVD($A \in \mathbb{R}^{m \times n}$)		
2:	$r := \operatorname{Rank}(A)$	
	$(V, \Lambda) := Diagonalize(A^{\top}A)$	\triangleright Sorted so that $\Lambda_{11} \ge \cdots \ge \Lambda_{nn}$
4:	Unpack $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$	
5:	$Unpack \Lambda := \begin{bmatrix} \Lambda_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix}$ $\Sigma_r := \Lambda_r^{1/2}$ $Pack \Sigma := \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$ $U_r := AV_r \Sigma_r^{-1}$	
6:	$\Sigma_r := \Lambda_r^{1/2}$	
7:	Pack $\Sigma := egin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$	
8:	$U_r := A V_r \Sigma_r^{-1}$	
9:	$U := \text{ExtendBasis}(U_r, \mathbb{R}^m)$	
10:	return (U, Σ, V)	
11: end function		



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