Discussion 14A

The following notes are useful for this discussion: Note 18.

1. Jacobians and Linear Approximation

Recall that for a scalar-valued function $f(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ with vector-valued arguments, we can linearize the function at $(\vec{x}_\star, \vec{y}_\star)$:

$$\widehat{f}(\vec{x},\vec{y}) = f(\vec{x}_\star,\vec{y}_\star) + \sum_{i=1}^{n} \frac{\partial f(\vec{x}_\star,\vec{y}_\star)}{\partial x_i}(x_i - x_{i,\star}) + \sum_{j=1}^{k} \frac{\partial f(\vec{x}_\star,\vec{y}_\star)}{\partial y_j}(y_j - y_{j,\star}). \tag{1}$$

In order to simplify this equation, we can define the following two vector quantities:

$$J_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \tag{2}$$

$$J_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_k} \end{bmatrix} \tag{3}$$

(a) When the function $\vec{f}(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a *vector* (rather than a scalar), we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them as above:

$$\hat{\vec{f}}(\vec{x},\vec{y}) = \begin{bmatrix} \hat{f}_1(\vec{x},\vec{y}) \\ \hat{f}_2(\vec{x},\vec{y}) \\ \vdots \\ \hat{f}_m(\vec{x},\vec{y}) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}_\star,\vec{y}_\star) + J_{\vec{x}} f_1 \cdot (\vec{x} - \vec{x}_\star) + J_{\vec{y}} f_1 \cdot (\vec{y} - \vec{y}_\star) \\ f_2(\vec{x}_\star,\vec{y}_\star) + J_{\vec{x}} f_2 \cdot (\vec{x} - \vec{x}_\star) + J_{\vec{y}} f_2 \cdot (\vec{y} - \vec{y}_\star) \\ \vdots \\ f_m(\vec{x}_\star,\vec{y}_\star) + J_{\vec{x}} f_m \cdot (\vec{x} - \vec{x}_\star) + J_{\vec{y}} f_m \cdot (\vec{y} - \vec{y}_\star) \end{bmatrix} \tag{4}$$

We can rewrite this in a clean way with the *Jacobian* of a vector-valued function:

$$J_{\vec{x}} \vec{f} = \begin{bmatrix} J_{\vec{x}} f_1 \\ J_{\vec{x}} f_2 \\ \vdots \\ J_{\vec{x}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \tag{5}$$

and similarly

$$J_{\vec{y}} \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \tag{6}$$

Then, the linearization becomes

$$\hat{\vec{f}}(\vec{x},\vec{y}) = \vec{f}(\vec{x}_\star,\vec{y}_\star) + J_{\vec{x}} \vec{f}(\vec{x}_\star,\vec{y}_\star) \cdot (\vec{x} - \vec{x}_\star) + J_{\vec{y}} \vec{f}(\vec{x}_\star,\vec{y}_\star) \cdot (\vec{y} - \vec{y}_\star). \tag{7}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$. Find $J_{\vec{x}} \vec{f}$, applying the definition above.

Solution: Here, we have

$$J_{\vec{x}} \vec{f} = \begin{bmatrix} 2x_1x_2 & x_1^2 \\ x_2^2 & 2x_1x_2 \end{bmatrix}. \tag{8}$$

(b) **Evaluate the approximation of $\vec{f}$ using $\vec{x}_\star = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ at the point $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$, and compare with $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$. Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$.**

Solution: Let $\delta = 0.01$. The true value is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} (2+\delta)^2(3+\delta) \\ (2+\delta)(3+\delta)^2 \end{bmatrix} = \begin{bmatrix} 12 + 16\delta + 7\delta^2 + \delta^3 \\ 18 + 21\delta + 8\delta^2 + \delta^3 \end{bmatrix}. \tag{9}$$

On the other hand, our approximation is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) \approx \vec{f}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + \begin{bmatrix} 12 & 4 \\ 9 & 12 \end{bmatrix} \cdot \begin{bmatrix} \delta \\ \delta \end{bmatrix} = \begin{bmatrix} 12 + 16\delta \\ 18 + 21\delta \end{bmatrix}. \tag{10}$$

Again, our approximation essentially removes the higher order terms of δ. When we plug in $\delta = 0.01$, we have

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} 12.160701 \\ 18.210801 \end{bmatrix} \tag{11}$$

and our approximation is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} 12.16 \\ 18.21 \end{bmatrix}. \tag{12}$$

(c) Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x}, \vec{y}) = \vec{x}\vec{y}^\top \vec{w}$. **Find $J_{\vec{x}}\vec{f}$ and $J_{\vec{y}}\vec{f}$.**

Solution: Here, recall that

$$\vec{f} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 & y_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 w_1 + x_1 y_2 w_2 \\ x_2 y_1 w_1 + x_2 y_2 w_2 \end{bmatrix}. \tag{13}$$

Then,

$$J_{\vec{x}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} y_1 w_1 + y_2 w_2 & 0 \\ 0 & y_1 w_1 + y_2 w_2 \end{bmatrix} \tag{14}$$

and

$$J_{\vec{y}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} x_1 w_1 & x_1 w_2 \\ x_2 w_1 & x_2 w_2 \end{bmatrix}. \tag{15}$$

We can also write

$$J_{\vec{x}}\vec{f} = \vec{y}^\top \vec{w} \cdot I \tag{16}$$

and

$$J_{\vec{y}}\vec{f} = \vec{x}\vec{w}^\top, \tag{17}$$

which can be derived by noticing that $\vec{y}^\top \vec{w} = \vec{w}^\top \vec{y}$.

(d) **(PRACTICE)** Continuing the above part, **find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.**

Solution: We have

$$\vec{f}(\vec{x}, \vec{y}) \approx \vec{f}(\vec{x}_\star, \vec{y}_\star) + J_{\vec{x}}\vec{f} \cdot (\vec{x} - \vec{x}_\star) + J_{\vec{y}}\vec{f} \cdot (\vec{y} - \vec{y}_\star) \tag{18}$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 - 1 \\ y_2 - 1 \end{bmatrix} \tag{19}$$

Let's do an approximation of $\vec{f}\left(\begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix}, \begin{bmatrix} 1+\delta_3 \\ 1+\delta_4 \end{bmatrix}\right)$, then,

$$\vec{f}\left(\begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix}, \begin{bmatrix} 1+\delta_3 \\ 1+\delta_4 \end{bmatrix}\right) \approx \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 3 + 3\delta_1 + 2\delta_3 + \delta_4 \\ 3 + 3\delta_2 + 2\delta_3 + \delta_4 \end{bmatrix}. \tag{20}$$

We can compare with the true value

$$\begin{aligned} \vec{f}\left(\begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix}, \begin{bmatrix} 1+\delta_3 \\ 1+\delta_4 \end{bmatrix}\right) &= \begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix} \begin{bmatrix} 1+\delta_3 & 1+\delta_4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \end{bmatrix} (3 + 2\delta_3 + \delta_4) \\ &= \begin{bmatrix} 3 + 3\delta_1 + 2\delta_3 + \delta_4 + 2\delta_1\delta_3 + \delta_1\delta_4 \\ 3 + 3\delta_2 + 2\delta_3 + \delta_4 + 2\delta_2\delta_3 + \delta_2\delta_4 \end{bmatrix}, \end{aligned} \tag{21}$$

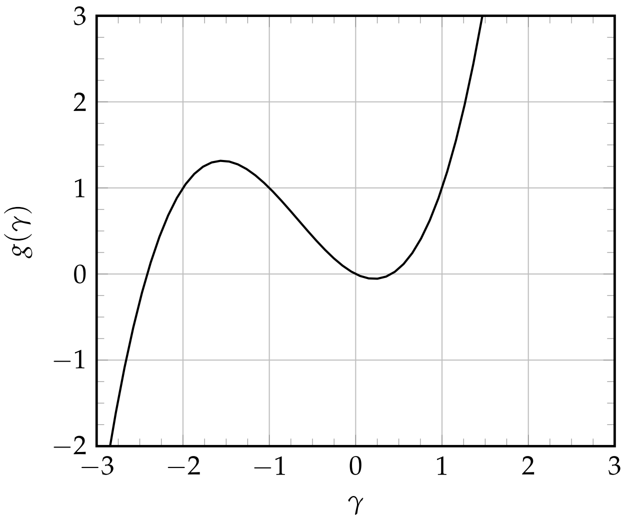
and we see that our approximation removes the second order δ terms $\delta_1\delta_3$, $\delta_1\delta_4$, $\delta_2\delta_3$ and $\delta_2\delta_4$.

2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix}-2\beta(t)+\gamma(t)\\ g(\gamma(t))+u(t)\end{bmatrix} = \vec{f}(\vec{x}(t),u(t)) \tag{22}$$

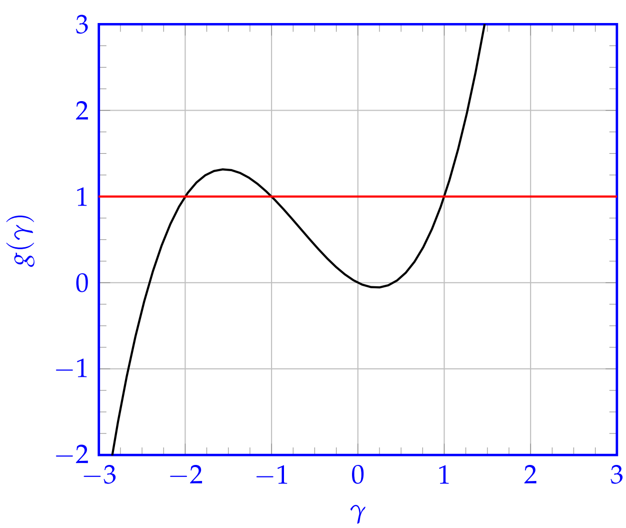
where $\vec{x}(t) = \begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix}$ and $g(\cdot)$ is a nonlinear function with the following graph:



The $g(\cdot)$ is the only nonlinearity in this system. We want to linearize this entire system around a operating point/equilibrium. Any point $\vec{x}_\star$ is an operating point if $\vec{f}(\vec{x}_\star(t), u_\star(t)) = \vec{0}$.

(a) **If we have fixed $u_\star(t) = -1$, what values of γ and β will ensure $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{f}(\vec{x}(t), u(t)) = \vec{0}$?**

Solution: To find the equilibrium point, we'll start by finding the values for which $g(\gamma) + u^\star = g(\gamma) - 1 = 0$. In other words, we need to find values of γ such that $g(\gamma) = 1$. Although we don't have an equation for $g(\gamma)$, we can still find these points graphically, by using our graph. If we add a horizonal line at $g(\gamma) = 1$, we get the following:



Having done this, it looks like we'll have $f_2(\vec{x}, u^\star) = g(\gamma) - u^\star = 0$ for $\gamma = -2, \gamma = -1$, and $\gamma = 1$.

Now we just need to find an β that sets $f_1(\vec{x}, u^\star) = -2\beta + \gamma = 0$ for each of these. Setting $\beta = \frac{1}{2} \cdot \gamma$ will do this.

With that, we have our three equilibrium points, namely

$$\vec{x}_1^\star = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \qquad \vec{x}_2^\star = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix} \qquad \vec{x}_3^\star = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}. \tag{23}$$

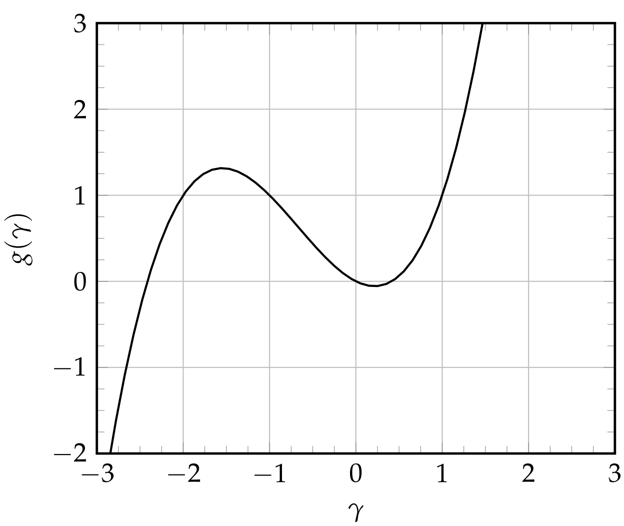
(b) Now that you have the three operating points, **linearize the system about the operating point $(\vec{x}_3^\star, u_\star)$ (that which has the largest value for γ).** Specifically, what we want is as follows. Let $\vec{\delta x}_i(t) = \vec{x}(t) - \vec{x}_i^\star$ for $i = 1, 2, 3$, and $\delta u(t) = u(t) - u_\star$. We can in principle write the linearized system for each operating point in the following form:

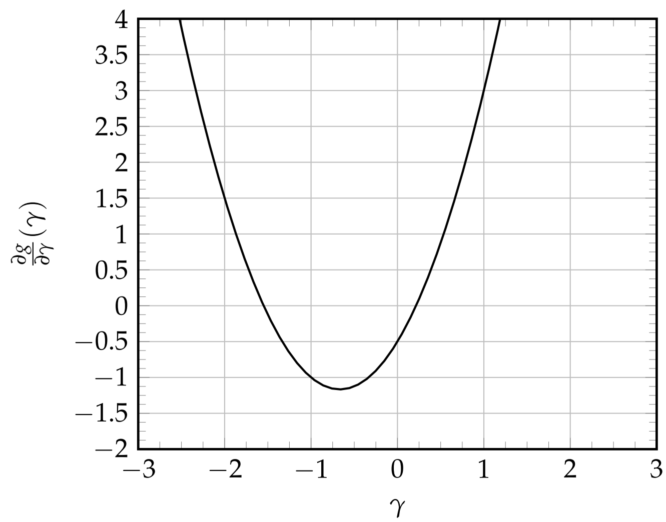
$$\text{(linearization about } (\vec{x}_i^\star, u_\star)) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}_i(t) = A_i \delta\vec{x}_i(t) + B_i \delta u(t) + \vec{w}_i(t) \tag{24}$$

where $\vec{w}_i(t)$ is a disturbance that also includes the approximation error due to linearization.

For this part, **find** A_3 **and** B_3.

We have provided below the function $g(\gamma)$ and its derivative $\frac{\partial g}{\partial \gamma}$.





Solution: To linearize the system, we need to compute the two Jacobians

$$J_{\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial \beta} & \frac{\partial f_1}{\partial \gamma} \\ \frac{\partial f_2}{\partial \beta} & \frac{\partial f_2}{\partial \gamma} \end{bmatrix} \tag{25}$$

$$J_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u}, \end{bmatrix} \tag{26}$$

and evaluate them at the operating points that we found in the previous part. The Jacobian matrices evaluated at the operating points will be the A_i and B_i matrices.

If we work out the partial derivatives, we get

$$\frac{\partial f_1}{\partial \beta} = \frac{\partial}{\partial \beta}(-2\beta + \gamma) = -2 \tag{27}$$

$$\frac{\partial f_1}{\partial \gamma} = \frac{\partial}{\partial \gamma}(-2\beta + \gamma) = 1 \tag{28}$$

$$\frac{\partial f_2}{\partial \beta} = \frac{\partial}{\partial \beta}(g(\gamma) + u) = 0 \tag{29}$$

$$\frac{\partial f_2}{\partial \gamma} = \frac{\partial}{\partial \gamma}(g(\gamma) + u) = \frac{\partial g}{\partial \gamma} \tag{30}$$

$$\frac{\partial f_1}{\partial u} = \frac{\partial}{\partial u}(-2\beta + \gamma) = 0 \tag{31}$$

$$\frac{\partial f_2}{\partial u} = \frac{\partial}{\partial u}(g(\gamma) + u) = 1 \tag{32}$$

which gives

$$J_{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & \frac{\partial g}{\partial \gamma} \end{bmatrix} \tag{33}$$

$$J_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{34}$$

It turns out that the only part of $J_{\vec{x}}$ and J_u that depends on the operating point is $\partial g / \partial \gamma$, and we can read these off of the given graph. The relevant values are

$$\left. \frac{\partial g}{\partial \gamma} \right|_{\gamma=-2} = 1.5 \tag{35}$$

$$\left.\frac{\partial g}{\partial \gamma}\right|_{\gamma=-1} = -1 \tag{36}$$
$$\left.\frac{\partial g}{\partial \gamma}\right|_{\gamma=1} = 3, \tag{37}$$

which correspond to $\vec{x}_1^\star$,$\vec{x}_2^\star$, and $\vec{x}_3^\star$, respectively. Finally, this gives

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1.5 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{38}$$
$$A_2 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{39}$$
$$A_3 = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{40}$$

(c) **Which of the operating points are stable? Which are unstable?**

Solution: To assess the stability or instability of each operating point, we need to find the eigenvalues of each linearization. Since A_1, A_2, and A_3 are all upper triangular, their eigenvalues are just the two entries along their diagonals. So, the linearization will be stable if both diagonal entries are negative (remember, these are continuous-time systems), and unstable if they aren't both negative. This means that:

- $\vec{x}_1^\star$ is unstable, since the eigenvalues of A_1 are -2 and 1.5;
- $\vec{x}_2^\star$ is stable, since the eigenvalues of A_2 are -2 and -1;
- $\vec{x}_3^\star$ is unstable, since the eigenvalues of A_3 are -2 and 3.

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