Homework 7

This homework is due on Friday, October 14, 2022, at 11:59PM. Self-grades and HW re-submissions are due on the following Friday, October 21, 2022, at 11:59PM.

1. Existence and uniqueness of solutions to differential equations

When doing circuits or systems analysis, we sometimes model our system via a differential equation, and would often like to solve it to get the system trajectory. To this end, we would like to verify that a solution to our differential equation exists and is unique, so that our model is physically meaningful. There is a general approach to doing this, which is demonstrated in this problem.

We would like to show that there is a unique function $x\colon \mathbb{R} \to \mathbb{R}$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{1}$$

$$x(0) = x_0. \tag{2}$$

In order to do this, we will first verify that a solution x_d exists. To show that x_d is the unique solution, we will take an arbitrary solution y and show that $x_d(t) = y(t)$ for every t.

(a) First, let us show that a solution to our differential equation exists. **Verify that** $x_d(t) := x_0 \mathrm{e}^{\alpha t}$ **satisfies eq. (1) and eq. (2).**

(b) Now, let us show that our solution is unique. As mentioned before, suppose $y\colon \mathbb{R} \to \mathbb{R}$ also satisfies eq. (1) and eq. (2).

We want to show that $y(t) = x_d(t)$ for all t. Our strategy is to show that $\frac{y(t)}{x_d(t)} = 1$ for all t.

However, this particular differential equation poses a problem: if $x_0 = 0$, then $x_d(t) = 0$ for all t, so that the quotient is not well-defined. To patch this method, we would like to avoid using any function with x_0 in the denominator. One way we can do this is consider a modification of the quotient $\frac{y(t)}{x_d(t)} = \frac{y(t)}{x_0 \mathrm{e}^{\alpha t}}$; in particular, we consider the function $z(t) := \frac{y(t)}{\mathrm{e}^{\alpha t}}$.

Show that $z(t) = x_0$ **for all** t**, and explain why this means that** $y(t) = x_d(t)$ **for all** t**.**

(HINT: Show first that $z(0) = x_0$ and then that $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$. Argue that these two facts imply that $z(t) = x_0$ for all t. Then show that this implies $y(t) = x_d(t)$ for all t.)

(HINT: Remember that we said y is any solution to eq. (1) and eq. (2), so we only know these properties of y. If you need something about y to be true, see if you can show it from eq. (1) and eq. (2).)

(HINT: When taking $\frac{\mathrm{d}}{\mathrm{d}t}z(t)$, remember to use the quotient rule, along with what we know about y.)

2. Simple Scalar Differential Equations Driven by an Input

In this question, we will show the existence and uniqueness of solutions to differential equations with inputs. In particular, we consider the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{3}$$

$$x(0) = x_0 \tag{4}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a known function of time. Feel free to assume u is "nice" in the sense that it is integrable, continuous, and differentiable with bounded derivative – basically, let u be nice enough that all the usual calculus theorems work.

(a) We will first demonstrate the existence of a solution to eqs. (3) and (4).

Define $x_d\colon \mathbb{R} \to \mathbb{R}$ by

$$x_d(t) := \mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau \tag{5}$$

Show that x_d satisfies eqs. (3) and (4).

(HINT: When showing that x_d satisfies eq. (3), one possible approach to calculate the derivative of the integral term is to use the fundamental theorem of calculus and the product rule.)

(b) Now, we will show that x_d is the unique solution to eqs. (3) and (4).

Suppose that $y\colon \mathbb{R} \to \mathbb{R}$ also satisfies eqs. (3) and (4). **Show that $y(t) = x_d(t)$ for all t.**

(HINT: This time, show that $z(t) := y(t) - x_d(t) = 0$ for all t. Do this by showing that $z(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = \lambda z(t)$, then use the uniqueness theorem for homogeneous first-order linear differential equations from the last problem. Note that the specific form of $x_d(t)$ in eq. (5) is irrelevant for the solution and should not be used.)

(c) Extend the solution in eq. (5) to the diagonal, vector differential equation case. **Namely, show that the solution for**

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \Lambda\vec{x}(t) + \vec{b}u(t) \tag{6}$$

is given by

$$\vec{x}(t) = \mathrm{e}^{\Lambda t}\vec{x}(0) + \int_0^t \mathrm{e}^{\Lambda(t-\tau)}\vec{b}u(\tau)\,\mathrm{d}\tau \tag{7}$$

where $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^n$, $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix, $\vec{b} \in \mathbb{R}^n$, and $u(t) : \mathbb{R} \to \mathbb{R}$. For notational convenience, we define $\mathrm{e}^{\Lambda x} = \begin{bmatrix} \mathrm{e}^{\lambda_1 x} & & & \\ & \mathrm{e}^{\lambda_2 x} & & \\ & & \ddots & \\ & & & \mathrm{e}^{\lambda_n x} \end{bmatrix}$ where $\Lambda := \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$.

(HINT: You may use the fact that, if $\vec{z}(t) : \mathbb{R} \to \mathbb{R}^n$ and $\vec{z}(t) := \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$, it is the case that $\int_a^b \vec{z}(t)\,\mathrm{d}t = \begin{bmatrix} \int_a^b z_1(t)\,\mathrm{d}t \\ \int_a^b z_2(t)\,\mathrm{d}t \\ \vdots \\ \int_a^b z_n(t)\,\mathrm{d}t \end{bmatrix}$.)

(HINT: Consider breaking down the diagonal vector differential equation case into n different scalar cases. How can you combine the results from eq. (5) with this?)

(d) **Extend the result from the previous part for an arbitrary vector differenial equation given by**

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{8}$$

where A is a *diagonalizable* matrix (but not necessarily a *diagonal* matrix). You may assume that A can be diagonalized as $A = V\Lambda V^{-1}$. For notational convenience, you may want to define $\widetilde{\vec{b}} = V^{-1}\vec{b}$.

3. Eigenvectors and Diagonalization

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define V to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

Show that $AV = V\Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

(b) **Argue that V is invertible, and therefore**

$$A = V\Lambda V^{-1}. \tag{9}$$

(HINT: What condition on a matrix's columns means that it would be invertible? It is fine to cite the appropriate result from 16A.)

(c) **Write Λ in terms of the matrices A, V, and V^{-1}.**

(d) A matrix A is deemed diagonalizable if there exists a square matrix U so that A can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix D.

Show that the columns of U must be eigenvectors of the matrix A, and that the entries of D must be eigenvalues of A.

(HINT: Recall the definition of an eigenvector (i.e., $A\vec{v} = \lambda\vec{v}$). Then, recall what $U^{-1}U$ is. Lastly, consider how matrix multiplication works column-wise.)

The previous part shows that the *only* way to diagonalize A is using its eigenvalues/eigenvectors. Now we will explore a payoff for diagonalizing A – an operation that diagonalization makes *much* simpler.

(e) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A \cdot A}_{k \text{ times}} \tag{10}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Conclude that A^k is diagonalizable.

4. Vector Differential Equations

Note: it's recommended to finish the previous question (Eigenvectors and Diagonalization) before this problem.

Consider a system of ordinary differential equations that can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) \tag{11}$$

where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are scalar functions of time t, and $A \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix with constant coefficients. We call eq. (11) a vector differential equation.

(a) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (11).

Consider a second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}y(t)}{\mathrm{d}t} + by(t) = 0, \tag{12}$$

where $a, b \in \mathbb{R}$.

Write this differential equation in the form of (eq. (11)), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.

(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (14) without having to take a second derivative, and instead just taking the first derivative of something.)

(b) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues have a solution in the general form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{13}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

Now let $a = -1$ and $b = -2$ in eq. (12), i.e.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \frac{\mathrm{d}y(t)}{\mathrm{d}t} - 2y(t) = 0, \tag{14}$$

Solve eq. (14) with the initial conditions $y(0) = 1$, $\frac{\mathrm{d}y}{\mathrm{d}t}(0) = 1$, using the general form in eq. (13).

(HINT: You get two equations using the initial conditions above. How many unknowns are here?) (HINT: Given your specific choice of x_1 and x_2 in part (f), how many unknowns are there really?)

5. System Identification

You are given a discrete-time system as a black box. You don't know the specifics of the system but you know that it takes one scalar input and has two states that you can observe. You assume that the system is linear and of the form

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{w}[i], \tag{15}$$

where $\vec{w}[i]$ is an external small unknown disturbance, $u[i]$ is a scalar input, and

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad x[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}. \tag{16}$$

You want to identify the system parameters (a_1, a_2, a_3, a_4, b_1 and b_2) from measured data. However, you can only interact with the system via a black box model, i.e., you can see the states $\vec{x}[t]$ and set the inputs $u[i]$ that allow the system to move to the next state.

(a) You observe that the system has state $\vec{x}[i] = \begin{bmatrix} x_1[i] & x_2[i] \end{bmatrix}^\top$ at time i. You pass input $u[i]$ into the black box and observe the next state of the system: $\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] & x_2[i+1] \end{bmatrix}^\top$.
Write scalar equations for the new states, $x_1[i+1]$ **and** $x_2[i+1]$. Write these equations in terms of the a_i, b_i, the states $x_1[i]$, $x_2[i]$ and the input $u[i]$. Here, assume that $\vec{w}[i] = \vec{0}$ (i.e., the model is perfect).

(b) Now we want to identify the system parameters. We observe the system at the start state $\vec{x}[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix}$. We can then input $u[0]$ and observe the next state $\vec{x}[1] = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix}$. We can continue this for a sequence of ℓ inputs.
Let us define an ℓ-length trajectory to be an initial condition $\vec{x}[0]$, an input sequence $u[0], \ldots, u[\ell-1]$, and the corresponding states that are produced by the system $x[1], \ldots, x[\ell]$. **Assuming that the model is perfect ($\vec{w}[i] = \vec{0}$), what is the minimum value of ℓ you need to identify the system parameters?**

(c) We now remove our assumption that $\vec{w} = 0$. We assume it is small, so the model is approximately correct and we have

$$\vec{x}[i+1] \approx A\vec{x}[i] + Bu[i]. \tag{17}$$

Say we feed in a total of 4 inputs $u[0], \ldots, u[3]$, and observe the states $\vec{x}[0], \ldots, \vec{x}[4]$. To identify the system we need to set up an approximate (because of potential, small, disturbances) matrix equation

$$DP \approx S \tag{18}$$

using the observed values above and the unknown parameters we want to find. Let our parameter vector be

$$P := \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \\ b_1 & b_2 \end{bmatrix} \tag{19}$$

Find the corresponding D and S to do system identification. Write both out explicitly.

(d) Now that we have set up $DP \approx S$, we can estimate a_0, a_1, a_2, a_3, b_0, and b_1. **Give an expression for the estimates of $\vec{p}_1$ and $\vec{p}_2$ (which are denoted $\hat{\vec{p}}_1$ and $\hat{\vec{p}}_2$ respectively) in terms of D and S.** Denote the columns of S as $\vec{s}_1$ and $\vec{s}_2$, so we have $S = [\vec{s}_1\ \vec{s}_2]$. Assume that the columns of D are linearly independent. *(HINT: Don't forget that D is not a square matrix. It is taller than it is wide.) (HINT: Can we split $DP = S$ into separate equations for p_1 and p_2?)*

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