# Homework 8

## This homework is due on Friday, October 21, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on Friday, October 28, 2022, at 11:59PM.

**1. Stability Criterion**

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

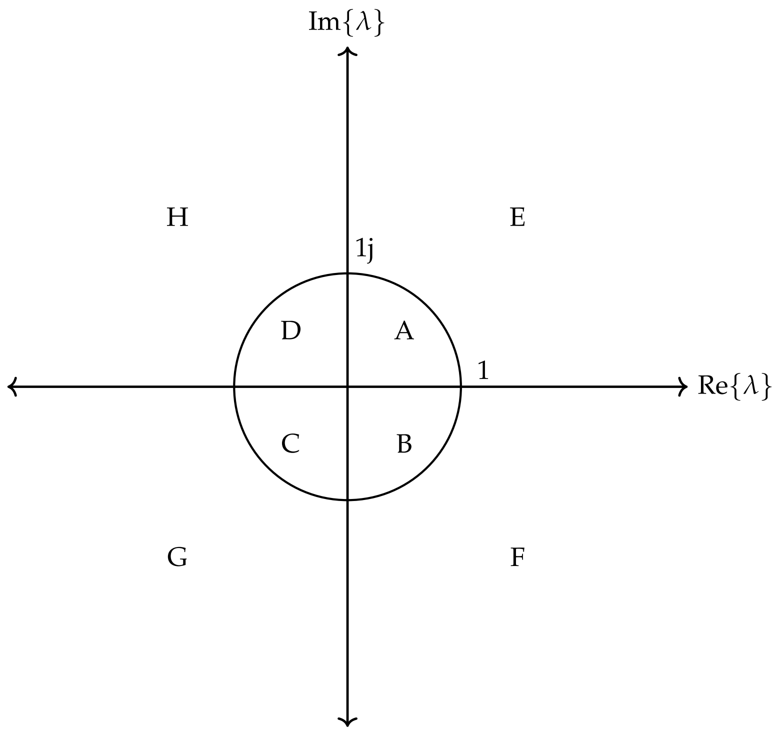



**Figure 1:** Complex plane divided into regions.

Consider the continuous-time system $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + v(t)$
and the discrete-time system $y[i+1] = \lambda y[i] + w[i]$. Here $v(t)$ and $w[i]$ are both disturbances to their respective systems.

**In which regions can the eigenvalue $\lambda$ be for the system to be *stable*? Fill out the table below to indicate *stable* regions.** Assume that the eigenvalue $\lambda$ does not fall directly on the boundary between two regions.

| | **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** |
|---|---|---|---|---|---|---|---|---|
| **Continuous Time System** $x(t)$ | ◯ | ◯ | ◯ | ◯ | ◯ | ◯ | ◯ | ◯ |
| **Discrete Time System** $y[i]$ | ◯ | ◯ | ◯ | ◯ | ◯ | ◯ | ◯ | ◯ |

**2. Bounded-Input Bounded-Output (BIBO) Stability**

BIBO stability is a system property where bounded inputs lead to bounded outputs. It's important because we want to certify that, provided our system inputs are bounded, the outputs will not "blow up". In this problem, we gain a better understanding of BIBO stability by considering some simple continuous and discrete systems, and showing whether they are BIBO stable or not.

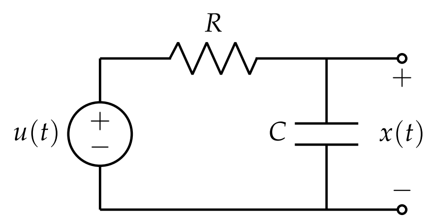
Recall that for the following simple scalar differential equation, we have the corresponding solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t) \qquad x(t) = \mathrm{e}^{at}x(0) + \int_0^t \mathrm{e}^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau\,. \tag{1}$$

And for the following discrete system, we have the corresponding solution:

$$x[i+1] = ax[i] + bu[i] \qquad x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k bu[i-1-k] \tag{2}$$

(a) Consider the circuit below with $R = 1\Omega$, $C = 0.5\text{F}$. Let $x(t)$ be the voltage over the capacitor.




This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -2x(t) + 2u(t) \tag{3}$$

Intuitively, we know that the voltage on the capacitor can never exceed the (bounded) voltage from the voltage source, so this system is BIBO stable. **Show that this system is BIBO stable, meaning that $x(t)$ remains bounded for all time if the input $u(t)$ is bounded. Equivalently, show that if we assume $|u(t)| < \epsilon,\ \forall t \geq 0$ and $|x(0)| < \epsilon$, then $|x(t)| < M,\ \forall t \geq 0$ for some positive constant $M$.** Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

*(HINT: eq. (1) may be useful. You may want to write the expression for $x(t)$ in terms of $u(t)$ and $x(0)$ and then take the norms of both sides to show a bound on $|x(t)|$. Remember that norm in 1D is absolute value. Some helpful formulas are $|ab| = |a||b|$, the triangle inequality $|a+b| \leq |a| + |b|$, and the integral version of the triangle inequality $\left|\int_a^b f(\tau)\,\mathrm{d}\tau\right| \leq \int_a^b |f(\tau)|\,\mathrm{d}\tau$, which just extends the standard triangle inequality to an infinite sum of terms.)*

(b) Assume $x(0) = 0$. **Show that the system eq. (1) is BIBO unstable when $a = \mathrm{j}2\pi$ by constructing a bounded input that leads to an unbounded $x(t)$.**

It can be shown that the system eq. (1) is unstable for any purely imaginary $a$ by a similar construction of a bounded input.

(c) Consider the discrete-time system and its solution in eq. (2). **Show that if $|a| > 1$, then even if $x[0] = 0$, a bounded input can result in an unbounded output, i.e. the system is BIBO unstable.** *(HINT: The formula for the sum of a geometric sequence may be helpful.)*

(d) Consider the discrete-time system

$$x[i+1] = -3x[i] + u[i]. \tag{4}$$

**Is this system stable or unstable? Give an initial condition $x(0)$ and a sequence of non-zero inputs for which the state $x[i]$ will always stay bounded.** *(HINT: See if you can find any input pattern that results in an oscillatory behavior.))*

**3. Eigenvalue Placement through State Feedback**

Consider the following discrete-time linear system:

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]. \tag{5}$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$.

(a) **Is this system controllable?**

(b) **Is this discrete-time linear system stable in open loop (without feedback control)?**

(c) Suppose we use state feedback of the form $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] = F\vec{x}[i]$.

**Find the appropriate state feedback constants, $f_1, f_2$ so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.**

**Contributors:**

- Sidney Buchbinder.
- Tanmay Gautam.
- Ashwin Vangipuram.
- Nathan Lambert.
- Anant Sahai.
- Sally Hui.
- Druv Pai.
- Varun Mishra.