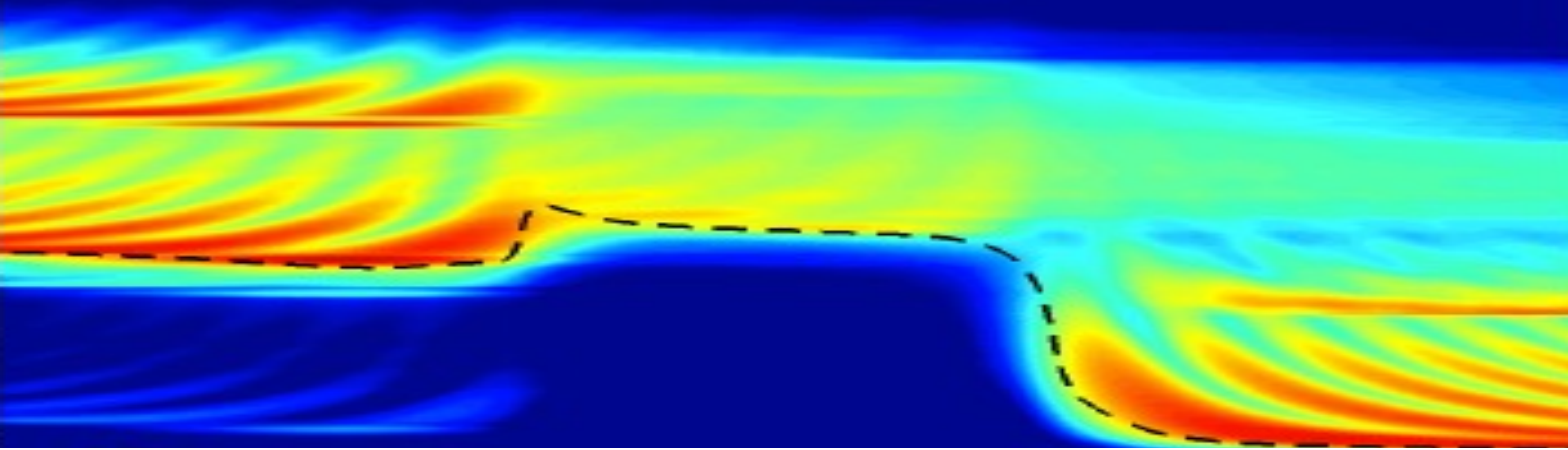


EECS 16B
Designing Information Devices and Systems II
Lecture 12

Prof. Yi Ma
Department of Electrical Engineering and Computer Sciences, UC Berkeley,
yima@eecs.berkeley.edu

Outline

- System Modeling
- Discretization (scalar and vector case)
- System Identification

System Modeling & Control

All autonomous intelligent (AI) systems rely on closed-loop learning and control:

External World (Real)

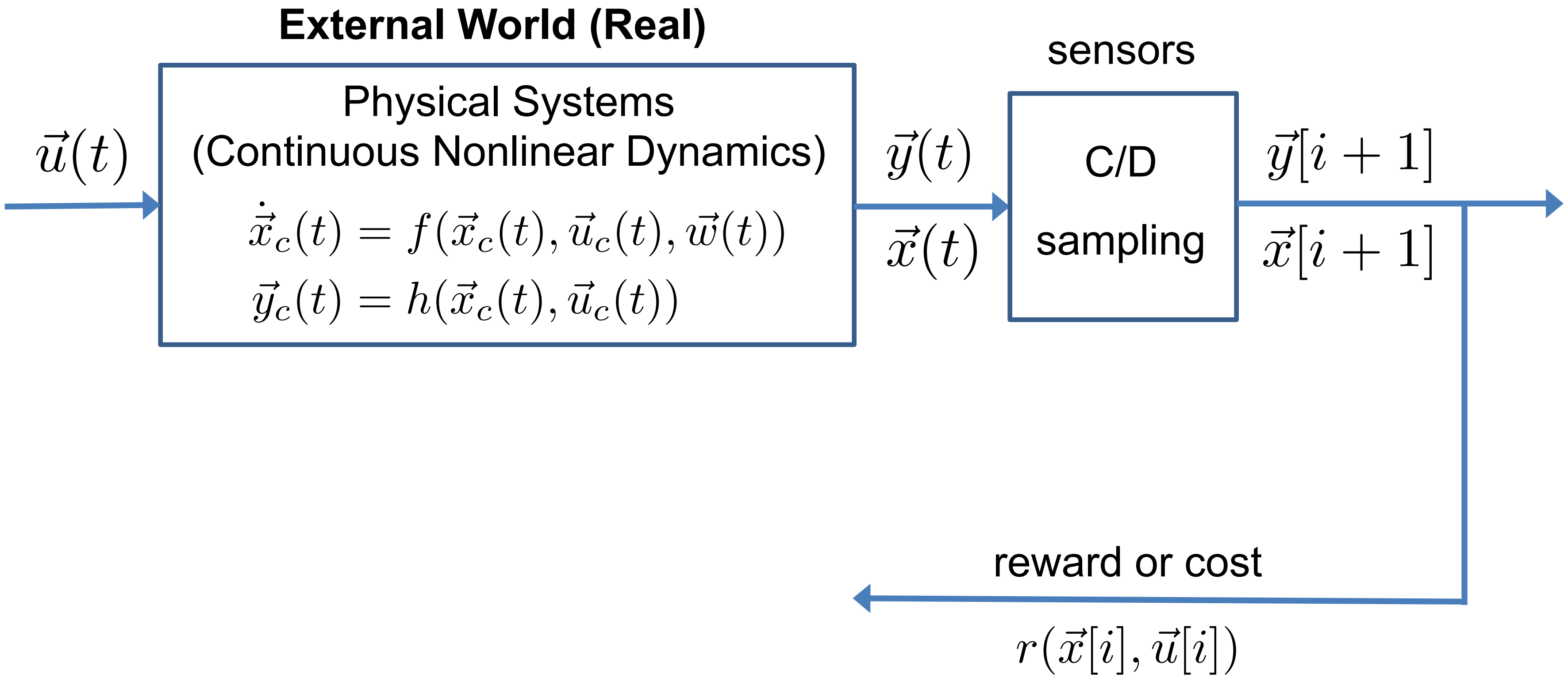
$\vec{u}(t)$ → Physical Systems (Continuous Nonlinear Dynamics) → $\vec{y}(t)$, $\vec{x}(t)$

Physical Systems
(Continuous Nonlinear Dynamics)

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
$$\vec{y}_c(t) = h(\vec{x}_c(t), \vec{u}_c(t))$$

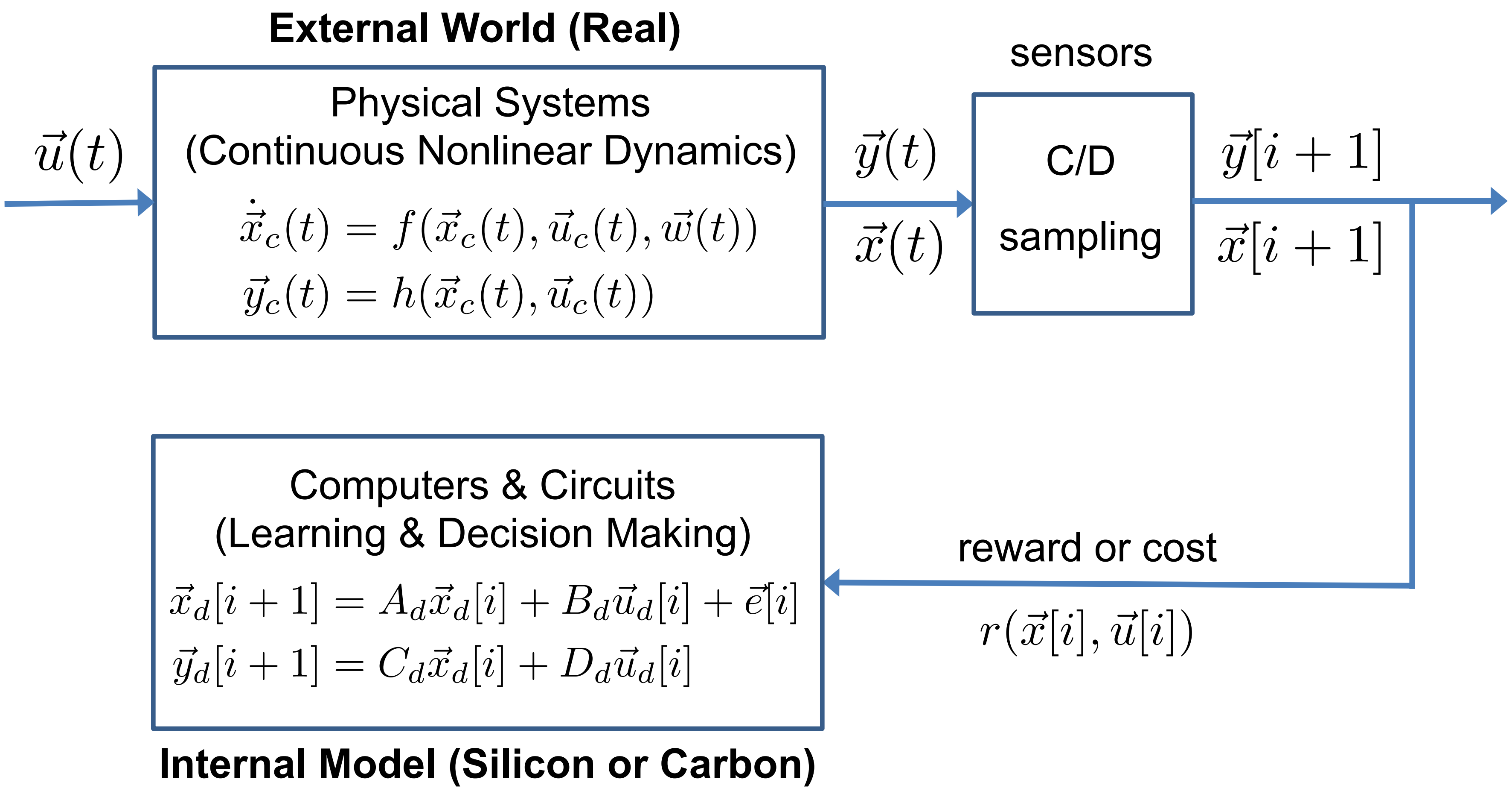
System Modeling & Control

All autonomous intelligent (AI) systems rely on closed-loop learning and control:



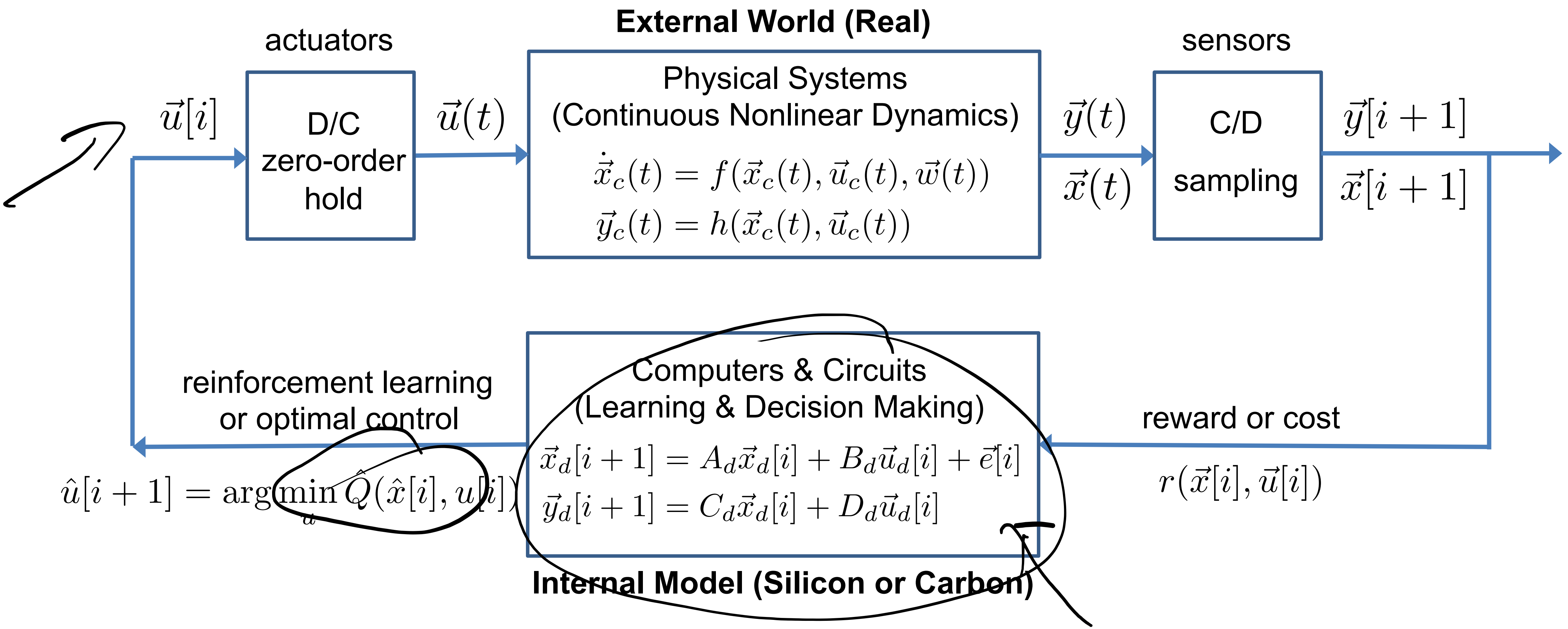
System Modeling & Control

All autonomous intelligent (AI) systems rely on closed-loop learning and control:

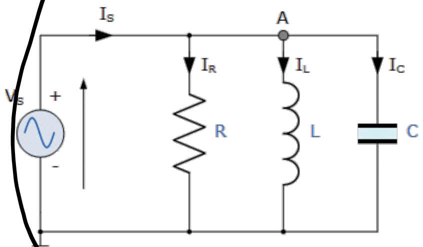


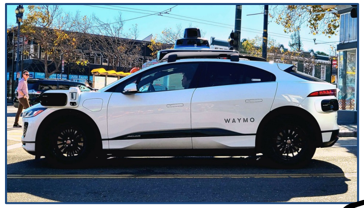
System Modeling & Control

All autonomous intelligent (AI) systems rely on closed-loop learning and control:



System Modeling





mathematical modeling
from first principles

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
$$\vec{y}_c(t) = h(\vec{x}_c(t), \vec{u}_c(t))$$

approximation
& linearization

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{n}(t)$$
$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$$

discretization
& digitization

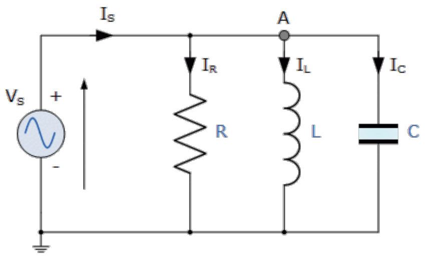
$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i]$$
$$\vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i]$$

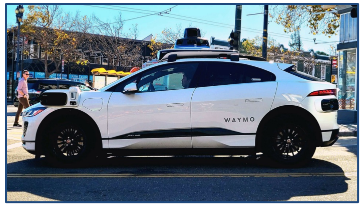
ODE: Newton's law.
PDE: Maxwell's equations
SDE: Markov processes

objectives of learning
* learn what is useful to predict.
* represent & organize

ReLU

System Modeling





mathematical modeling
from first principles

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
$$\vec{y}_c(t) = h(\vec{x}_c(t), \vec{u}_c(t))$$

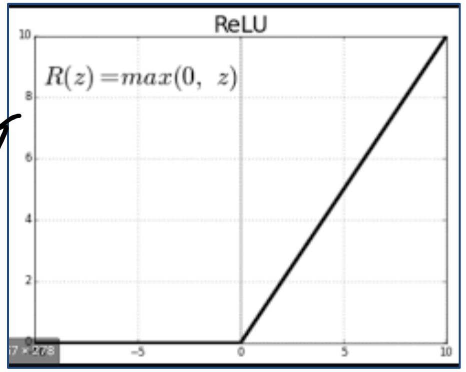
approximation
& linearization

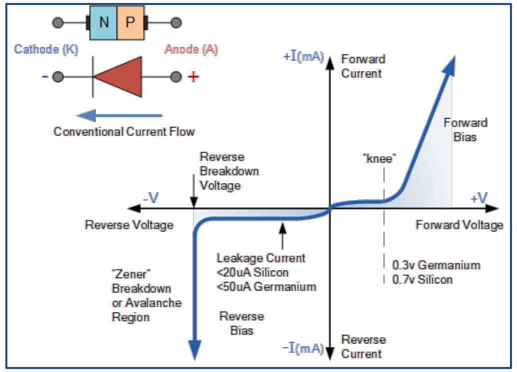
$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{n}(t)$$
$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$$

discretization
& digitization

$$\begin{cases} \vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i] \\ \vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i] \end{cases}$$

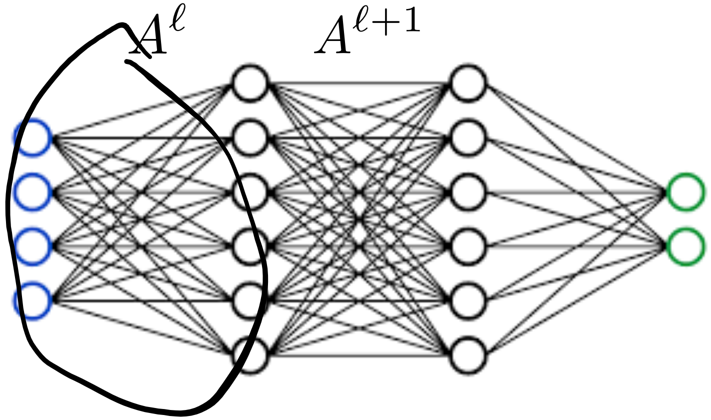
too simple?





$$\vec{x}_d[i+1] = \sigma_x(A_d\vec{x}_d[i] + B_d\vec{u}_d[i]) + \vec{e}[i]$$
$$\vec{y}_d[i+1] = \sigma_y(C_d\vec{x}_d[i] + D_d\vec{u}_d[i])$$

nonlinear activation
and concatenation

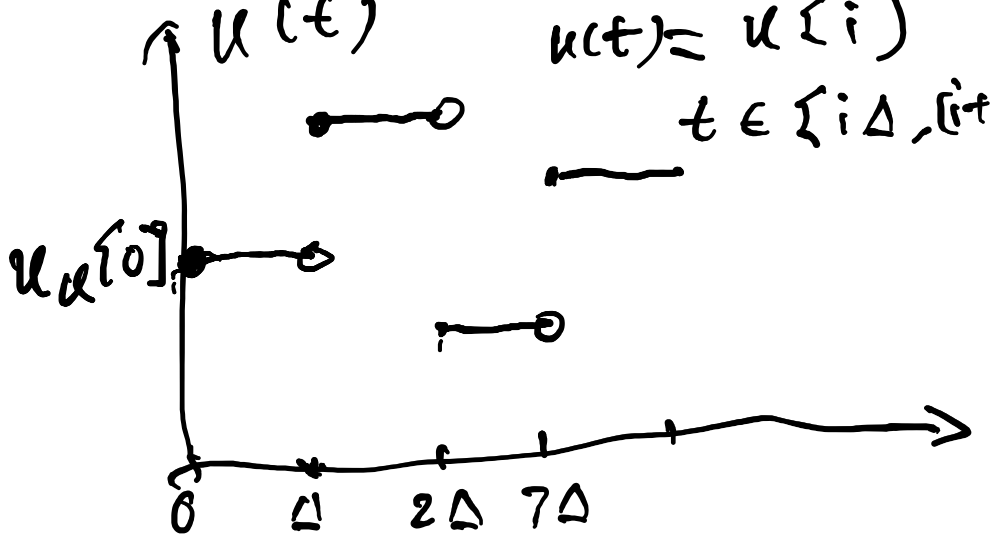


System Modeling: Discretization

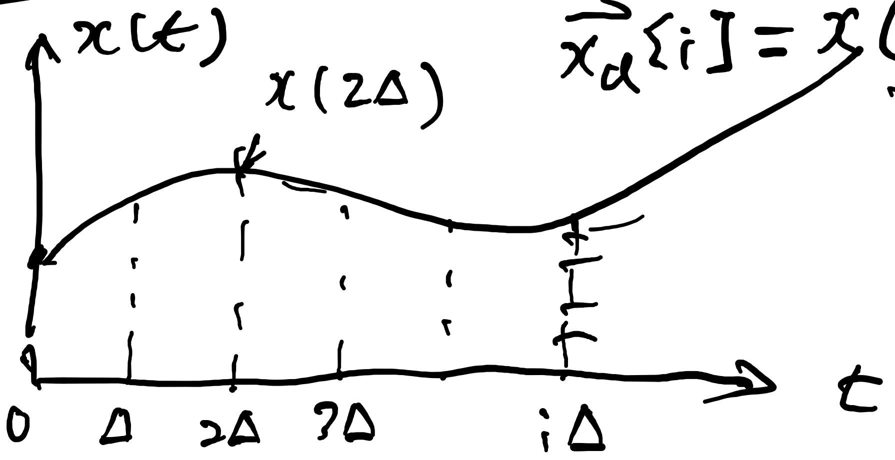
$\vec{u}_d[i]$ → D/C zero-order hold → $\vec{u}(t)$ → $\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$ → $\vec{x}(t)$ → C/D sampling → $\vec{x}_d[i+1]$

$u(t) = u[i]$

$t \in [i\Delta, (i+1)\Delta]$



$\vec{x}_d[i] = \vec{x}(i\Delta)$

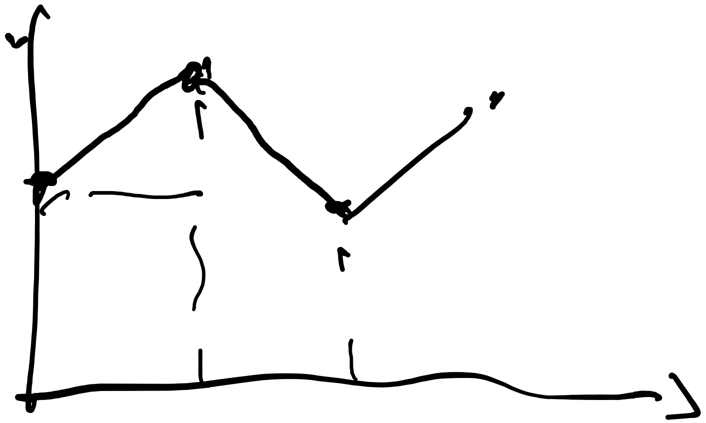


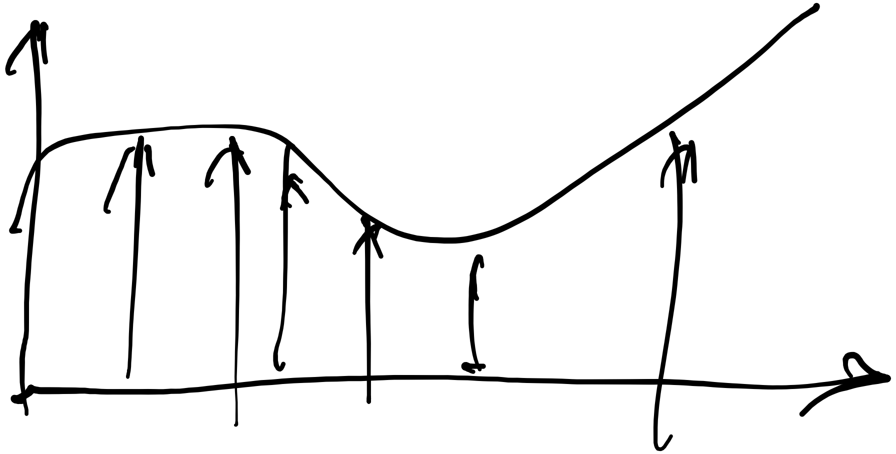
discretization

$\vec{u}_d[i]$ → $\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]$ → $\vec{x}_d[i+1]$

System Modeling: Discretization

$$\vec{u}_d[i] \longrightarrow \boxed{\text{D/C zero-order hold}} \xrightarrow{\vec{u}(t)} \boxed{\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)} \xrightarrow{\vec{x}(t)} \boxed{\text{C/D sampling}} \longrightarrow \vec{x}_d[i+1]$$





discretization

$$\vec{u}_d[i] \longrightarrow \boxed{\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]} \longrightarrow \vec{x}_d[i+1]$$

Discretization: Scalar Case

Scalar Case: $\boxed{\dot{x}(t) = ax(t) + bu(t)}$ → $\boxed{x_d[i+1] = A_d x_d[i] + B_d u_d[i]}$

$\rightarrow u(t) = u_d[i] \quad t \in [\underbrace{i\Delta}_{t_0}, \underbrace{(i+1)\Delta}_{t}]$

$$x(t) = e^{a(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} \underbrace{b\, u(\tau)}_{u_d[i]} d\tau$$

$t_0 = i\Delta, \quad t = (i+1)\Delta$

$$x_d[i+1] = x((i+1)\Delta) = e^{a\Delta} x_d[i] + \underbrace{\int_{i\Delta}^{(i+1)\Delta} e^{a(t-\tau)} d\tau}\, b u_d[i]$$

$$\boxed{x_d[i+1] = \underbrace{e^{a\Delta}}_{A_d} x_d[i] + \underbrace{\frac{e^{a\Delta}-1}{a} b}_{B_d} u_d[i]}$$

$$\boxed{\frac{e^{a\Delta}-1}{a}} \quad (= \Delta \text{ if } a = 0)$$

$a = 0$?

$\vec{x} \in \mathbb{R}^n$

Discretization: Vector Case

Vector Case: $\underbrace{\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)}$ $\rightarrow$ $\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]$

Diagonalizable: $\underline{\Lambda = V^{-1}AV}$ $\quad \vec{z} = V^{-1}\vec{x}$

$$V\Lambda = AV \qquad [v_1, v_2, \ldots, v_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} = A[v_1, v_2, \ldots, v_n]$$

$$\lambda_i v_i = A v_i$$

$$\vec{z} = \underline{V^{-1}\vec{x}} \qquad \vec{x} = \underline{V\vec{z}} \qquad \frac{d\vec{z}(t)}{dt} = V^{-1}\frac{d\vec{x}}{dt}$$

$$\frac{d\vec{z}(t)}{dt} = \underline{V^{-1}AV}\,\vec{z}(t) + \underline{V^{-1}B\vec{u}(t)}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad \tilde{u}(t)$$

Discretization: Vector Case

Vector Case: $\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$

$$\underline{\vec{x}_d[i+1]} = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]$$

Diagonalizable: $A = V^{-1}\Lambda V$

$$\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$$

$$\frac{d\vec{z}(t)}{dt} = \Lambda \vec{z}(t) + (V^{-1}B u(t))$$

$$\frac{dz_k(t)}{dt} = \lambda_k z_k(t) + \underbrace{(V^{-1}Bu(t))_k}_{\hat{u}_k(t)}$$

$$\tilde{u}_k[i] = \hat{u}_k(i\Delta)$$

$$z_{k,d}[i+1] = e^{\lambda_k \Delta} z_{k,d}[i] + \frac{e^{\lambda_k \Delta} - 1}{\lambda_k} \tilde{u}_k[i]$$

$$\vec{x}_d[i+1] = \underline{V}\,\vec{z}_d[i+1] \qquad \vec{z}_d[i] = V^{-1} x_d[i]$$

$A_{n\times n}$ $B_{n\times m}$ $\vec{u}(i\Delta)$

Discretization: Vector Case

Vector Case: $\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$ $\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]$

Diagonalizable: $A = V^{-1}\Lambda V$

$$\vec{x}_d[i+1] = \begin{bmatrix} e^{\lambda_1\Delta} & & \\ & \ddots & \\ & & e^{\lambda_n\Delta} \end{bmatrix} \vec{x}_d[i] + \begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} V^{-1}B\vec{u}_d[i]$$

$$\vec{x}_d[i+1] = \underbrace{V\left[\quad\right]V^{-1}}_{A_d}\vec{x}_d[i] + \underbrace{V\left[\quad\right]V^{-1}B}_{B_d}\vec{u}_d[i]$$

Discretization: General Case

General Case: $\dot{\vec{x}}(t) = \underline{A}\vec{x}(t) + \underline{B}\vec{u}(t)$ $\qquad$ $\vec{x}_d[i+1] = \underline{A_d}\vec{x}_d[i] + \underline{B_d}\vec{u}_d[i]$

$$\vec{x}(t) = e^{A(t-t_0)}\vec{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\vec{u}(\tau)d\tau$$

$$e^{\lambda t} = 1 + \lambda t + \frac{(\lambda t)^2}{2} + \frac{(\lambda t)^3}{6} + \cdots = \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!}$$

$$e^{At} = 1 + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$$

$$\vec{x}_d[i+1] = e^{A\Delta}\vec{x}_d[i] + \int_{i\Delta}^{(i+1)\Delta} e^{A(t-\tau)}Bd\tau\vec{u}_d[i]$$

$$A_d = e^{A\Delta}$$

$$B_d = (e^{A\Delta} - I)A^{-1}B$$

$$e^{a\Delta}$$

$$\frac{e^{a\Delta}-1}{a}$$

System Identification

Problem: consider the discrete linear time invariant system:

unknown

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

Given: observed inputs and outputs:

$$\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$$
$$\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots$$

Objective: learn the system parameters:

$$\vec{u}[i] \longrightarrow \boxed{A, B} \longrightarrow \vec{x}[i+1]$$

DNN.

$$u \longrightarrow \boxed{\text{black box}} \longrightarrow x$$

$$\vec{x}^T[i+1] = \begin{bmatrix} \vec{x}[i]^T & \vec{u}[i]^T \end{bmatrix} \begin{bmatrix} A^T \\ B^T \end{bmatrix} + \vec{e}^T[i]$$

$$y = Ax + b$$

Least Squares (Gauss 1809)

$$\vec{s} \in \mathbb{R}^a, \quad D \in \mathbb{R}^{a \times b}, \quad \vec{p} \in \mathbb{R}^b, \quad \vec{e} \in \mathbb{R}^a$$

$$\vec{s} = D \underbrace{\vec{p}}_{\text{unknown}} + \vec{e}, \quad \operatorname{rank}[D] = b \qquad D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_b]$$

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2$$

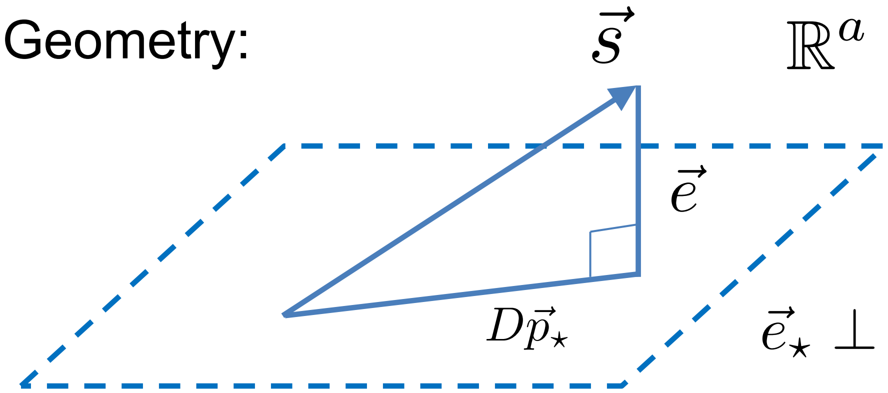
Least Squares (Gauss 1809)

$$\vec{s} \in \mathbb{R}^a, \quad D \in \mathbb{R}^{a \times b}, \quad \vec{p} \in \mathbb{R}^b, \quad \vec{e} \in \mathbb{R}^a$$

$$\vec{s} = D \underset{\text{unknown}}{\vec{p}} + \vec{e}, \quad \operatorname{rank}[D] = b \qquad D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_b]$$

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2$$

Geometry:



$$\vec{e}_\star \perp [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_b]$$

$$D^\top \vec{e} = D^\top (\vec{s} - D\vec{p}_\star) = \vec{0}$$