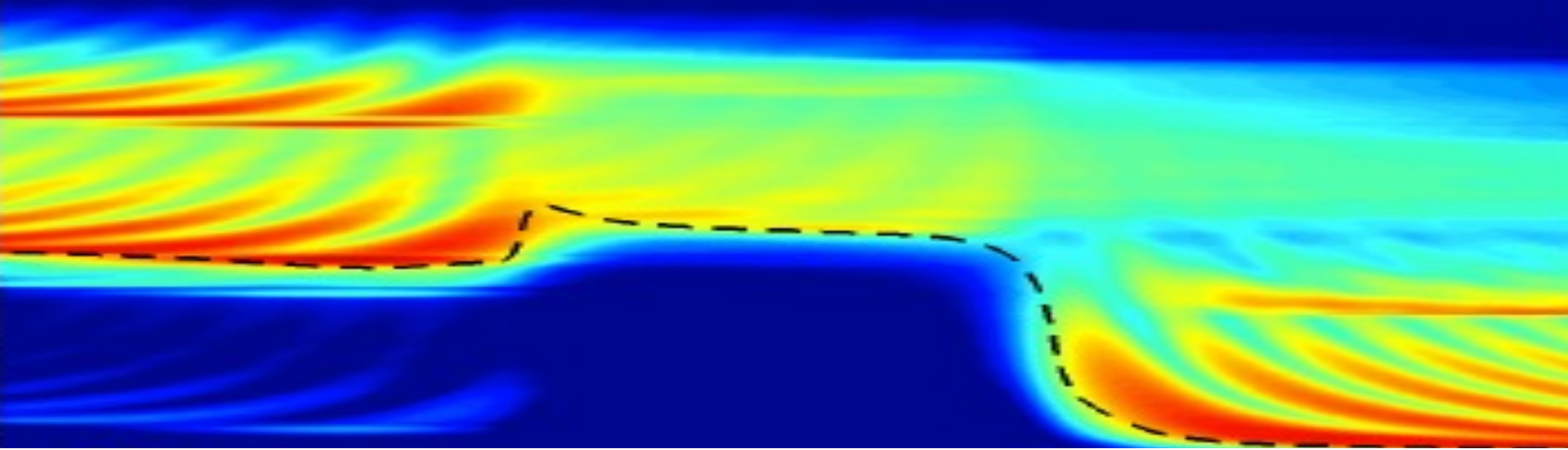

EECS 16B
Designing Information Devices and Systems II
Lecture 13

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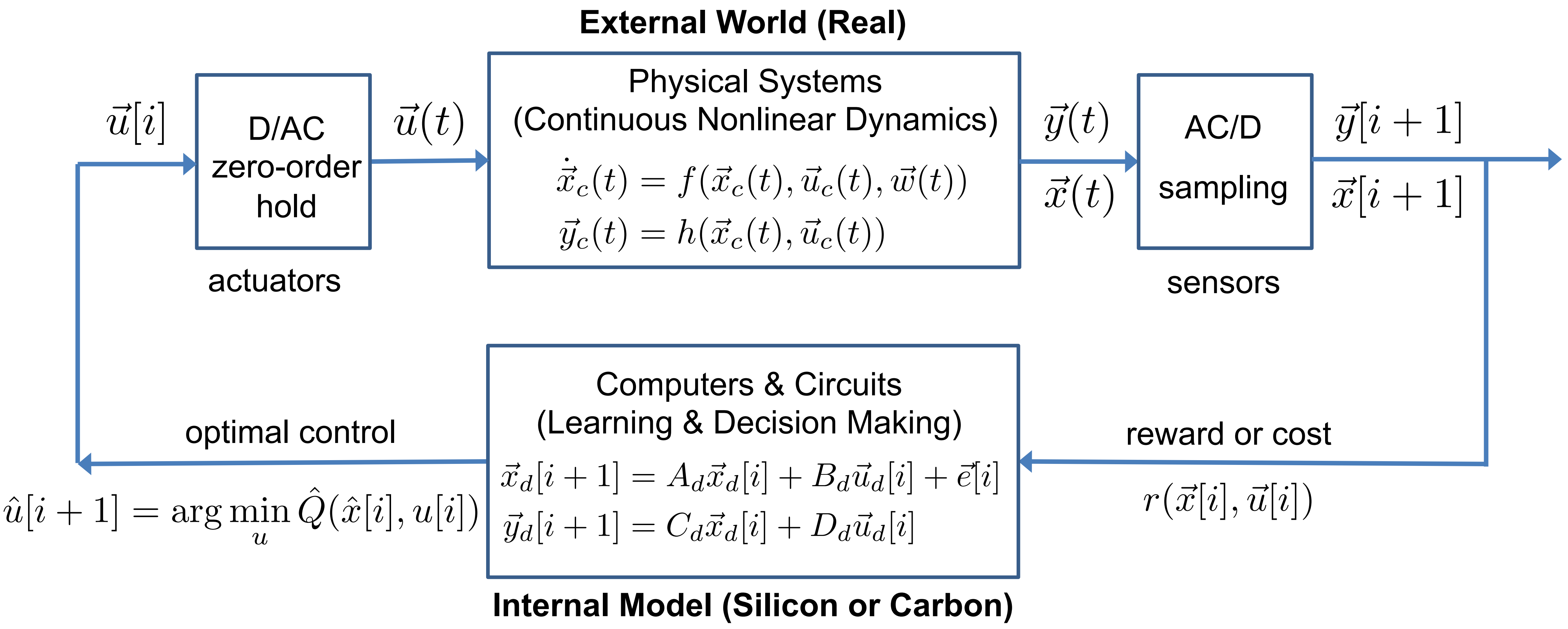
Outline

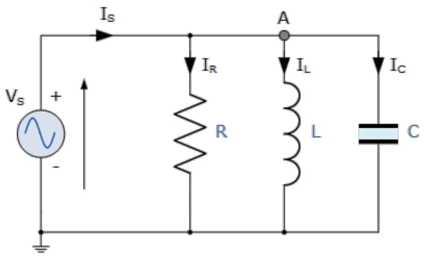
- System Modeling and Identification
- Least Squares and Extensions (vector and matrix case)
- System Stability (scalar case)

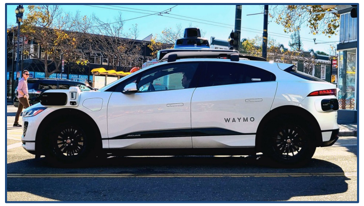
System Modeling & Control

All autonomous intelligent (AI) systems rely on closed-loop learning and control:



System Modeling & Identification





mathematical modeling
from first principles

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
$$\vec{y}_c(t) = h(\vec{x}_c(t), \vec{u}_c(t))$$

approximation
& linearization

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{n}(t)$$
$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$$

discretization
& digitization

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i]$$
$$\vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i]$$

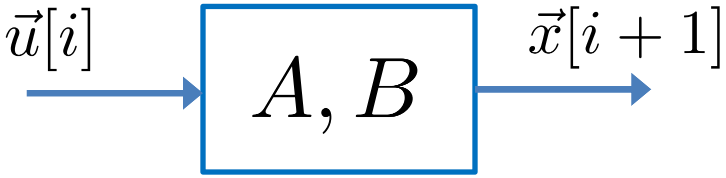
Problem: consider the discrete linear time invariant system:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

Given: observed inputs and outputs:

$$\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$$
$$\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots$$

Objective: learn the system parameters:



System Identification

Problem: consider the discrete linear time invariant system:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

Given: observed inputs and outputs:

$$\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$$
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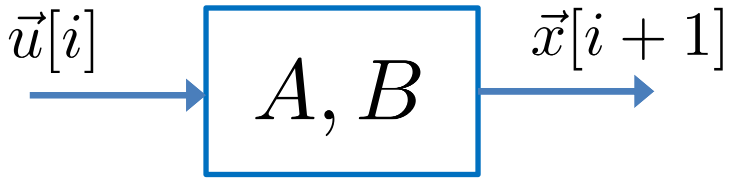
Objective: learn the system parameters:



Scalar Case:

$$x[i+1] = ax[i] + bu[i] + e[i]$$

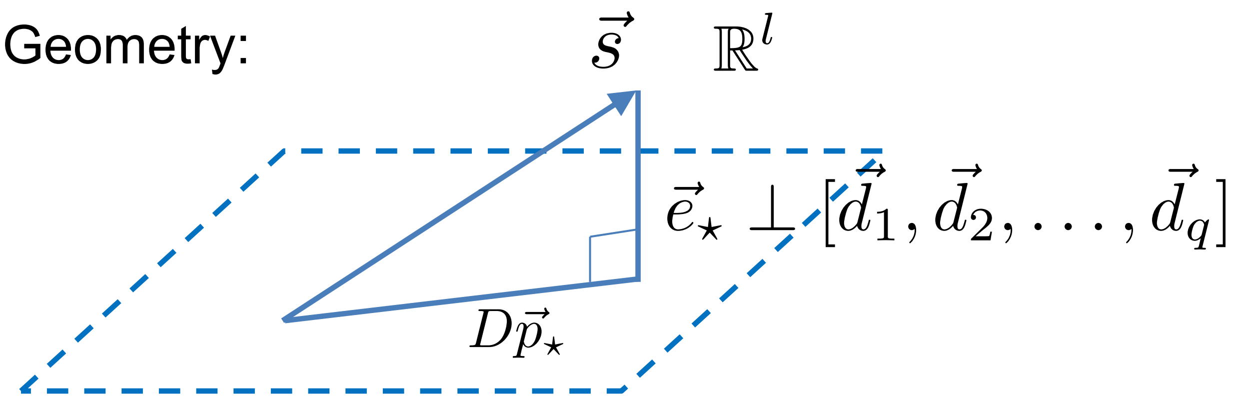
Least Squares (Gauss 1809)

$$\vec{s} \in \mathbb{R}^l, \quad D \in \mathbb{R}^{l \times q}, \quad \vec{p} \in \mathbb{R}^q, \quad \vec{e} \in \mathbb{R}^l$$

$$\vec{s} = D \underbrace{\vec{p}}_{\text{unknown}} + \vec{e}, \quad \text{rank}[D] = q \qquad D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_q]$$

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2$$

Geometry:



$$D^\top \vec{e}_\star = D^\top (\vec{s} - D\vec{p}_\star) = \vec{0}$$

Least Squares (Gauss 1809)

$$\vec{s} \in \mathbb{R}^l, \quad D \in \mathbb{R}^{l \times q}, \quad \vec{p} \in \mathbb{R}^q, \quad \vec{e} \in \mathbb{R}^l$$

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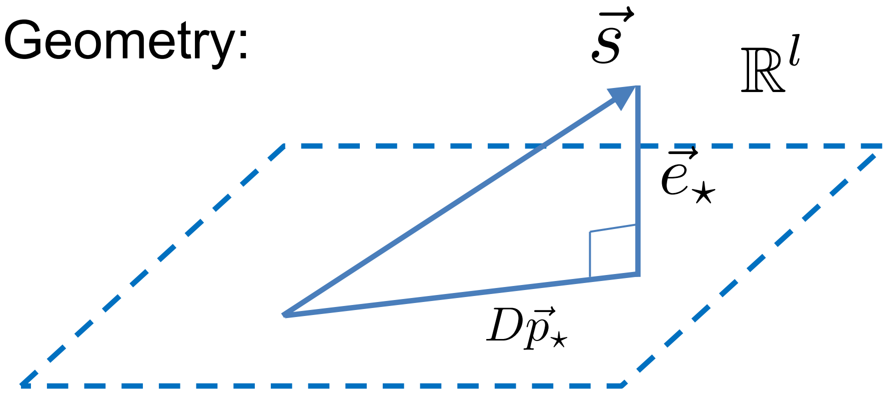
Geometry:



$$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_q] \perp \vec{e}_\star$$

$$D^\top \vec{e}_\star = D^\top (\vec{s} - D\vec{p}_\star) = \vec{0}$$

$$\min_{\vec{p}} f(\vec{p})$$

Algebra: $\dfrac{\partial \|\vec{s} - D\vec{p}\|_2^2}{\partial \vec{p}} \Big|_{\vec{p}_\star} = \vec{0}$

Least Squares: Some Extensions

$$\vec{s} \in \mathbb{R}^l, \quad D \in \mathbb{R}^{l \times q}, \quad \vec{p} \in \mathbb{R}^q, \quad \vec{e} \in \mathbb{R}^l \qquad \vec{s} = D \underbrace{\vec{p}}_{\text{unknown}} + \vec{e}$$

1. Over-determined $(l \geq q, \ \text{rank}[D] = q)$

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2 \quad = (D^\top D)^{-1} D^\top \vec{s}$$

2. Under-determined $(l < q, \ \text{rank}[D] = l)$

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{p}\|_2^2 \ \text{ s.t. } \ \vec{s} = D\vec{p} \quad = D^\top (DD^\top)^{-1} \vec{s}.$$

3. Ridge regression

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2 + \lambda \|\vec{p}\|_2^2 \quad = (D^\top D + \lambda I)^{-1} D^\top \vec{s}.$$

Least Squares: Matrix/Batch Case

$$S \in \mathbb{R}^{l \times m}, \quad D \in \mathbb{R}^{l \times q}, \quad P \in \mathbb{R}^{q \times m}, \quad E \in \mathbb{R}^{l \times m}$$

$$S = D \underbrace{P}_{\text{unknown}} + E, \quad \text{rank}[D] = q$$

$$P_\star = \arg\min_P \|S - DP\|_F^2$$

$$P_\star = (D^\top D)^{-1} D^\top S$$

System Identification

Problem: consider the discrete linear time invariant system:

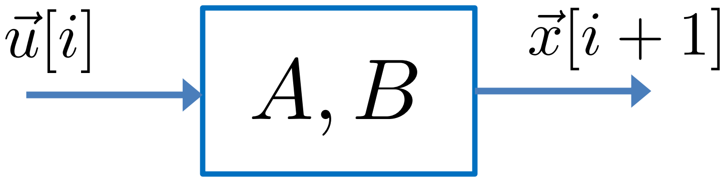
$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

Vector Case: $\vec{x} \in \mathbb{R}^n, \vec{e} \in \mathbb{R}^n, \vec{u} \in \mathbb{R}^m,$
$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}.$

Given: observed inputs and outputs:

$$\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$$
$$\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots$$

Objective: learn the system parameters:



System Identification

Problem: consider the discrete linear time invariant system:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

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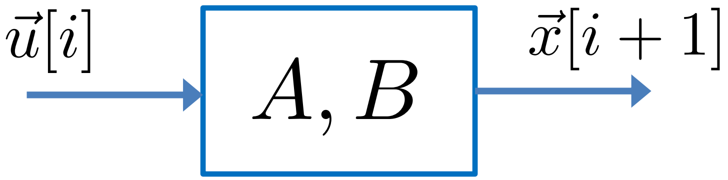


System Stability

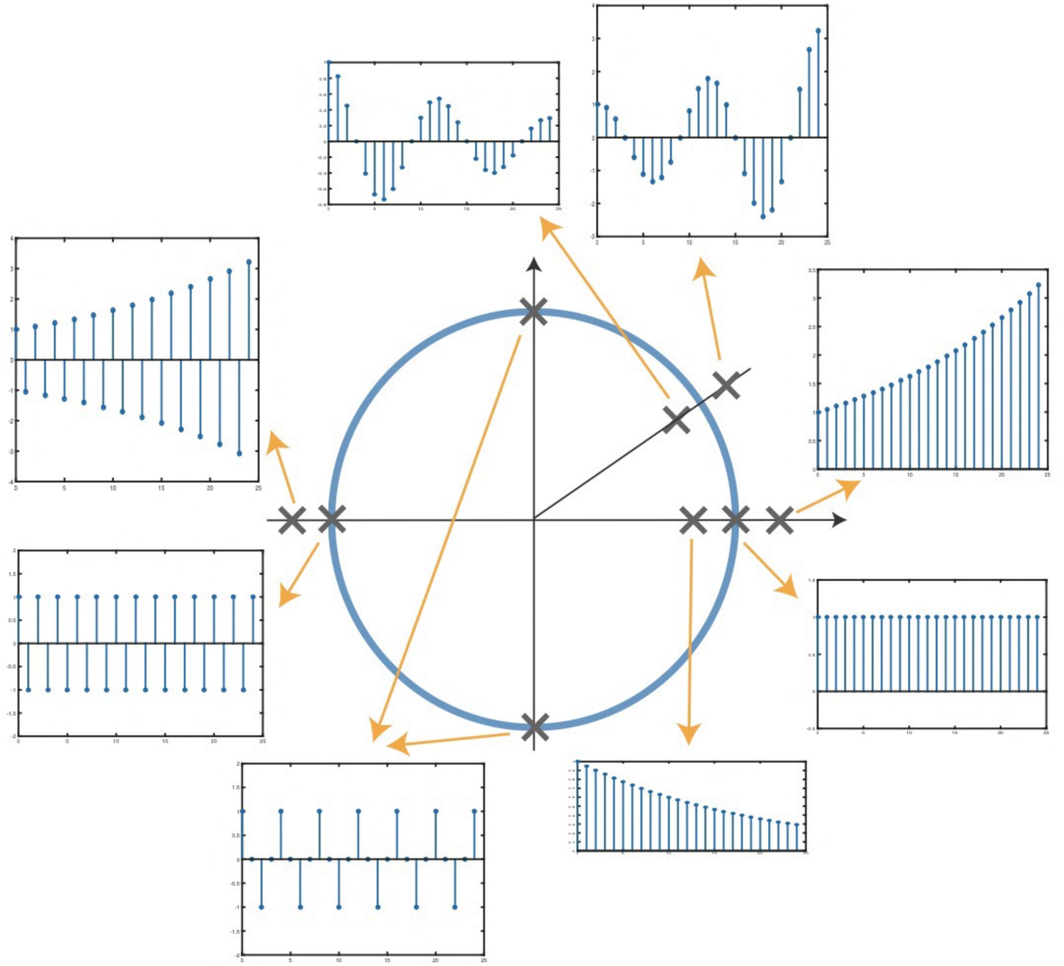
Scalar Case: $x[i+1] = \lambda x[i] + u[i] + e[i] \quad (\text{with } u[i] = 0)$

$x[i+1] = \lambda x[i] \quad (\text{with } \lambda > 1)$

$x[i+1] = \lambda x[i] \quad (\text{with } \lambda \leq 1)$

System Stability

Complex λ: $x[i+1] = \lambda x[i] \quad (\text{with } \lambda = |\lambda| e^{j\theta})$



System Stability

Critical Case $|\lambda| = 1$**:** $x[i+1] = \lambda x[i] + e[i]$

Bounded Input Bounded State Stability

Definition: We say a system is *bounded input bounded state (BIBS) stable* if its state stays bounded, $\forall i \ \|\vec{x}[i]\| \leq C$, for any initial condition, any bounded input, and bounded disturbance.

$$x[i+1] = \lambda x[i] + u[i] + e[i] \ \in \mathbb{R} \qquad \vec{x}[i+1] = A\vec{x}[i] + \vec{u}[i] + \vec{e}[i] \ \in \mathbb{R}^n$$

When is the above scalar system stable by this definition?

What about the vector case?