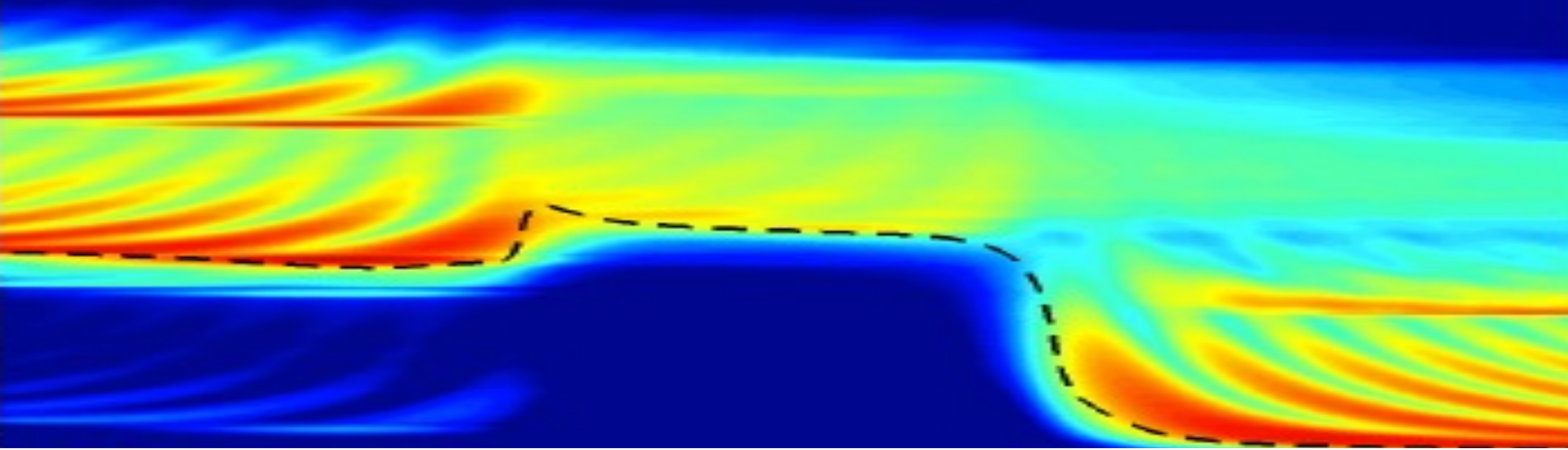


EECS 16B
Designing Information Devices and Systems II
Lecture 13

Prof. Yi Ma

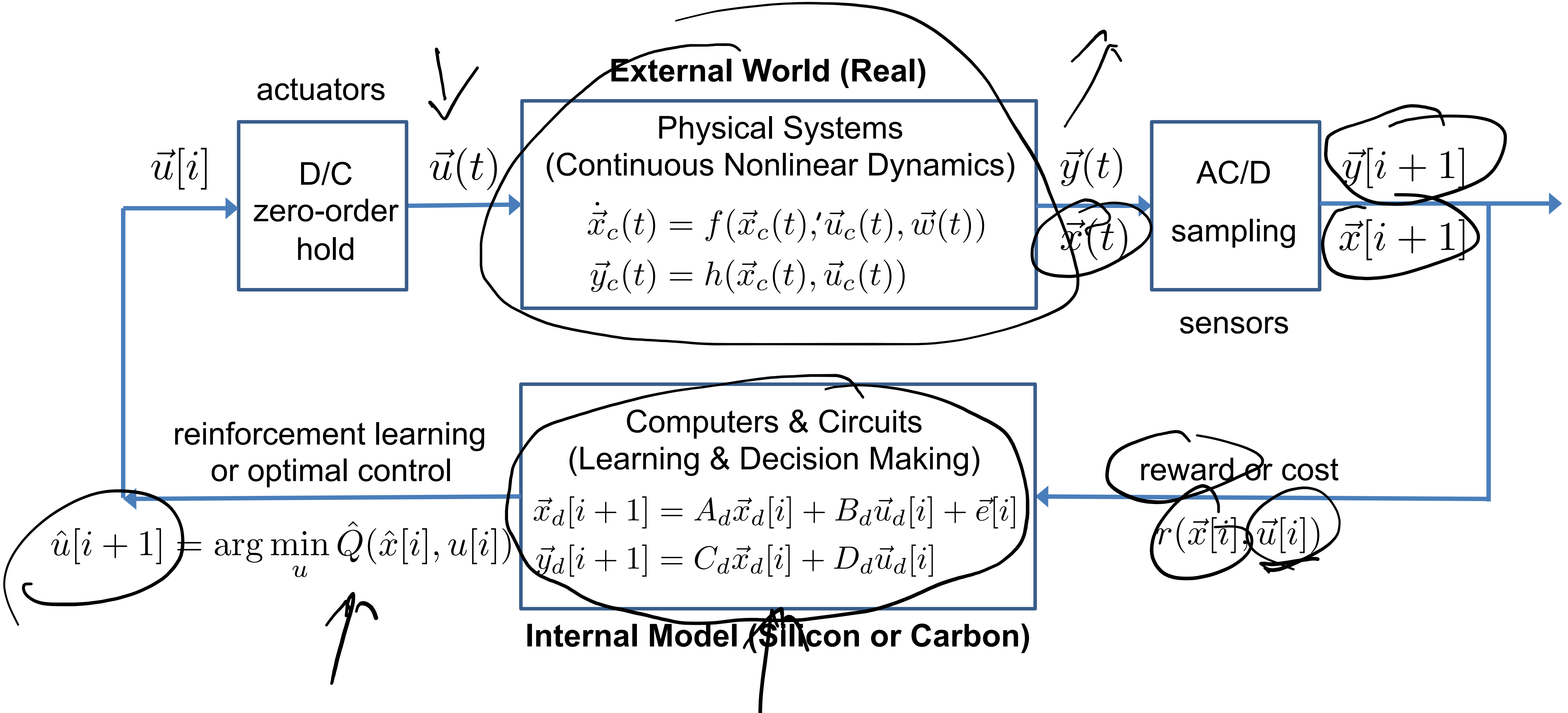
Department of Electrical Engineering and Computer Sciences, UC Berkeley,
yima@eecs.berkeley.edu

Outline

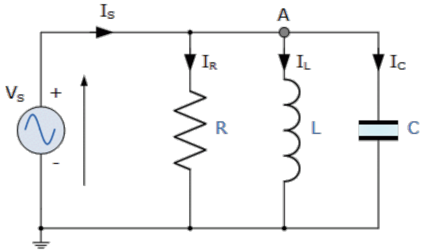
- System Modeling and Identification
- Least Squares and Extensions (vector and matrix case)
- System Stability (scalar case)

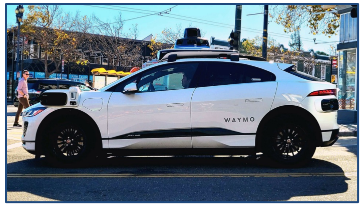
System Modeling & Control

All autonomous intelligent (AI) systems rely on closed-loop learning and control:



System Modeling & Identification





mathematical modeling from first principles

$$\dot{\vec{x}}_c(t) = f(\vec{x}_c(t), \vec{u}_c(t), \vec{w}(t))$$
$$\vec{y}_c(t) = h(\vec{x}_c(t), \vec{u}_c(t))$$

approximation & linearization

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{n}(t)$$
$$\vec{y}(t) = C\vec{x}(t) + D\vec{u}(t)$$

discretization & digitization

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i] + \vec{e}[i]$$
$$\vec{y}_d[i+1] = C_d\vec{x}_d[i] + D_d\vec{u}_d[i]$$

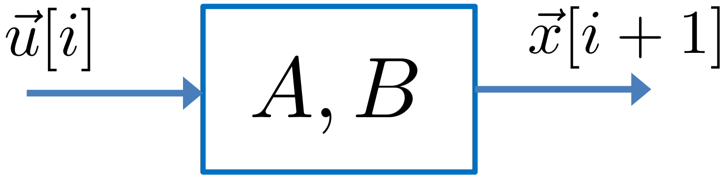
Problem: consider the discrete linear time invariant system:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

Given: observed inputs and outputs:

$$\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$$
$$\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots$$

Objective: learn the system parameters:



$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 7 & 5 \end{bmatrix}$$

System Identification

$l+2.$

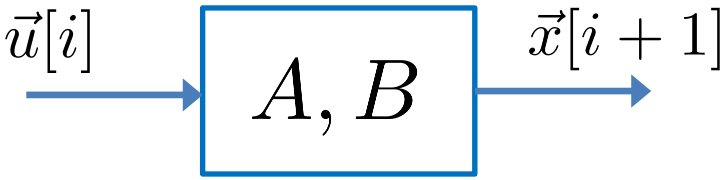
Problem: consider the discrete linear time invariant system:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

Given: observed inputs and outputs: l

$$\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$$
$$\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots$$

Objective: learn the system parameters:



Scalar Case: unknowns

$$x[i+1] = a x[i] + b u[i] + e[i]$$

$$x[1] = a x[0] + b u[0] + e[0]$$
$$x[2] = a x[1] + b u[1] + e[1]$$
$$\vdots$$
$$x[l] = a x[l-1] + b u[l-1] + e[l-1]$$

$$\begin{bmatrix} x[1] \\ \vdots \\ x[l] \end{bmatrix} = \begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \\ x[l-1], & u[l-1] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e[0] \\ e[1] \\ \vdots \\ e[l-1] \end{bmatrix}$$

$$\vec{s} = D\vec{p} + \vec{e}$$

Least Squares (Gauss 1809)

$\vec{s} \in \mathbb{R}^l, \quad D \in \mathbb{R}^{l \times q}, \quad \vec{p} \in \mathbb{R}^q, \quad \vec{e} \in \mathbb{R}^l$

$\vec{s} = D \underbrace{\vec{p}}_{\text{unknown}} + \underbrace{\vec{e}}_{\text{unknown}}, \quad \text{rank}[D] = q \quad D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_q]$

$\min \sum_{i=1}^{l} e_i^2 = \sum_{i=1}^{l} (s_i - d_i p)^2$

$e_i \sim$ small $(s_i = d_i p + e_i$

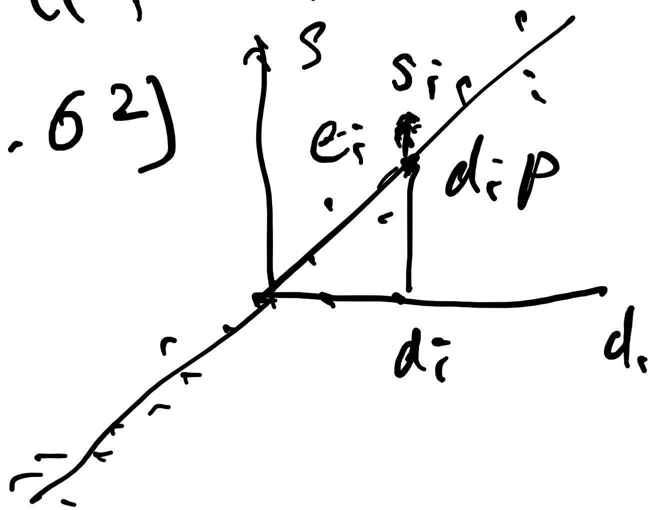
$e_i \sim N(0, \sigma^2)$

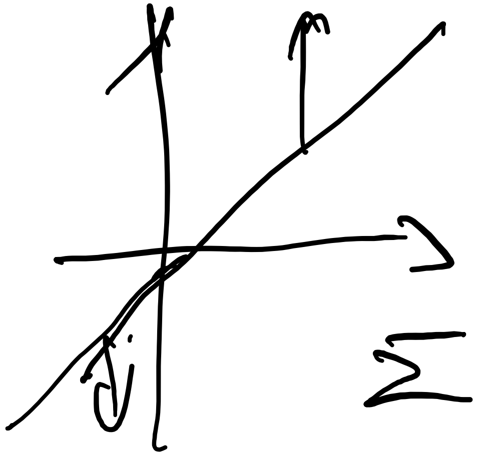
$\min \| \vec{s} - D\vec{p} \|_2^2$

$l \geq q$

$p_1 \vec{d}_1 + p_2 \vec{d}_2 + \cdots + p_q \vec{d}_q$

$\vec{e} = \vec{s} - D\vec{p}$





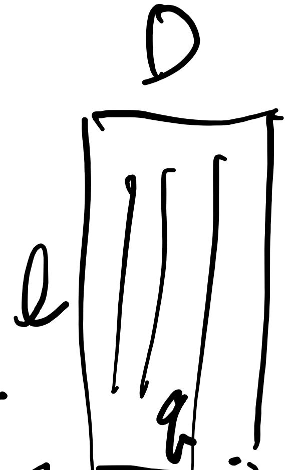
$\sum | s_i - d_i p |$

Least Squares (Gauss 1809)

$\vec{s} \in \mathbb{R}^l, \quad D \in \mathbb{R}^{l \times q}, \quad \vec{p} \in \mathbb{R}^q, \quad \vec{e} \in \mathbb{R}^l$

$$\vec{s} = D \underbrace{\vec{p}}_{\text{unknown}} + \underbrace{\vec{e}}_{\text{unknown}}, \quad \operatorname{rank}[D] = q$$

$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_q]$

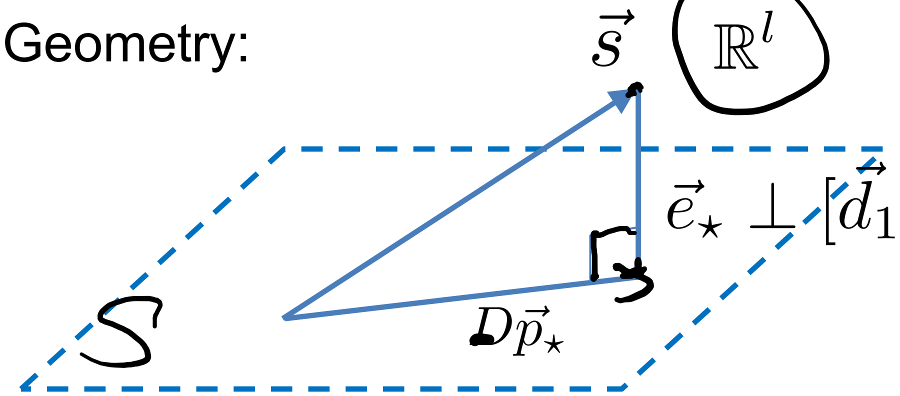


$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2$$

$\mathbb{R}^l$

$S = \operatorname{span}(D)$ – column space

Geometry:



$\vec{d}_i^{\,T} \vec{e}_* = 0$

$$\begin{bmatrix} \vec{d}_1^{\,T} \\ \vec{d}_q^{\,T} \end{bmatrix} \vec{e}_* = 0$$

$D^T D_{q \times q}$

$D^T(\vec{s} - D\vec{p}_*) = 0 \qquad D^T\vec{s} - D^T D\vec{p}_* = 0$

$$D^\top \vec{e}_\star = D^\top(\vec{s} - D\vec{p}_\star) = \vec{0} \Rightarrow$$

$$\vec{p}_* = (D^T D)^{-1} D^T \vec{s}.$$

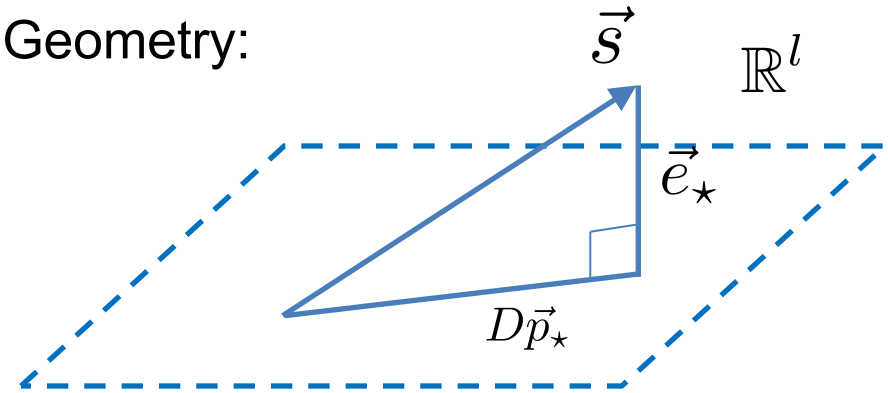
Least Squares (Gauss 1809)

$$\vec{s} \in \mathbb{R}^l, \quad D \in \mathbb{R}^{l\times q}, \quad \vec{p} \in \mathbb{R}^q, \quad \vec{e} \in \mathbb{R}^l$$

$$\vec{s} = D \underbrace{\vec{p}}_{\text{unknown}} + \vec{e}, \quad \operatorname{rank}[D] = q$$

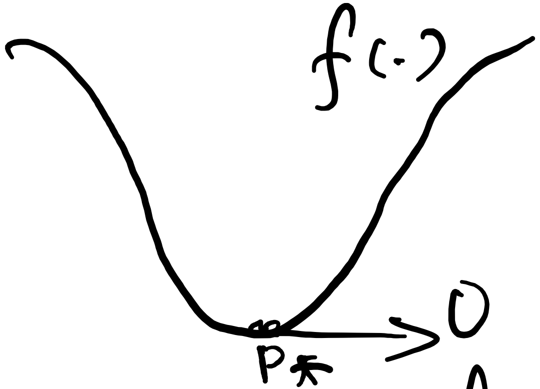
$$\vec{p}_\star = \arg\min_{\vec{p}} \underbrace{\|\vec{s} - D\vec{p}\|_2^2} = f(\vec{p})$$

Geometry:



$$D = [\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_q] \perp \vec{e}_\star$$

$$\underbrace{D^\top \vec{e}_\star = D^\top(\vec{s} - D\vec{p}_\star) = \vec{0}}$$



$$\min_{\vec{p}} f(\vec{p})$$

Algebra: $\dfrac{\partial\|\vec{s} - D\vec{p}\|_2^2}{\partial \vec{p}}\Big|_{\vec{p}_\star} = \vec{0}$

$$\frac{\partial f(\vec{p})}{\partial \vec{p}} = \frac{\partial(\vec{s} - D\vec{p})^T(\vec{s} - D\vec{p})}{\partial \vec{p}}$$

$$0 = -2D^T\vec{s} + 2D^TD\vec{p}_*$$

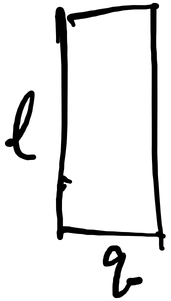
$$\boxed{\vec{p}_* = (D^TD)^{-1}D^T\vec{s}}$$

Least Squares: Some Extensions

$$\vec{s} \in \mathbb{R}^l, \quad D \in \mathbb{R}^{l\times q}, \quad \vec{p} \in \mathbb{R}^q, \quad \vec{e} \in \mathbb{R}^l \qquad \vec{s} = D \underset{\text{unknown}}{\vec{p}} + \vec{e}$$

1. Over-determined $(l \geq q, \ \mathrm{rank}[D] = q)$

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2 = (D^\top D)^{-1} D^\top \vec{s}$$



2. Under-determined $(l < q, \ \mathrm{rank}[D] = l)$

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{p}\|_2^2 \ \text{ s.t. } \ \vec{s} = D\vec{p} \quad = D^\top(DD^\top)^{-1}\vec{s}.$$

minimum energy

$\vec{s} = D(\vec{p} + \vec{n})$ $\quad \vec{n} \in \mathrm{Null}(D)$

$D\vec{n} = 0$

3. Ridge regression

$$\vec{p}_\star = \arg\min_{\vec{p}} \|\vec{s} - D\vec{p}\|_2^2 + \lambda\|\vec{p}\|_2^2 = \boxed{(D^\top D + \lambda I)^{-1} D^\top \vec{s}.}$$

Least Squares: Matrix/Batch Case

$$S \in \mathbb{R}^{l\times m}, \quad D \in \mathbb{R}^{l\times q}, \quad P \in \mathbb{R}^{q\times m}, \quad E \in \mathbb{R}^{l\times m}$$

$$S = D \underbrace{P}_{\text{unknown}} + \underbrace{E}_{\text{unknown}}, \quad \operatorname{rank}[D] = q$$

$[\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_m] = D\,[\vec{p}_1, \ldots, \vec{p}_m] + [\vec{e}_1, \ldots, \vec{e}_m]$

$\vec{s}_i = D\,\vec{p}_i + \vec{e}_i \quad \min \|\vec{e}_i\|_2^2$

$$P_\star = \arg\min_P \|S - DP\|_F^2$$

$\min \sum_{i=1}^{m} \|\vec{e}_i\|_2^2 = \|S - DP\|_F^2$

$$P_\star = (D^\top D)^{-1} D^\top S$$

$[\vec{p}_{1\star}, \vec{p}_{2\star} \ldots] = (D^T D)^{-1} D^T [\vec{s}_1, \vec{s}_2, \ldots]$

System Identification

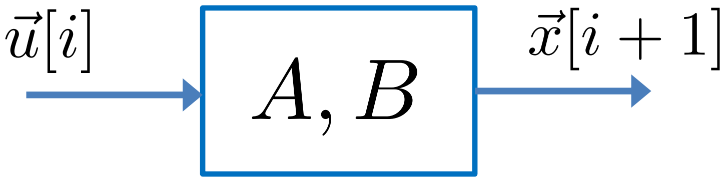
Problem: consider the discrete linear time invariant system:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i] \in \mathbb{R}^n$$

Given: observed inputs and outputs:

$$\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$$
$$\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots$$

Objective: learn the system parameters:



Vector Case: $\vec{x} \in \mathbb{R}^n, \vec{e} \in \mathbb{R}^n, \vec{u} \in \mathbb{R}^m,$
$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}.$

$$\vec{x}[1]^T = \vec{x}[0]^T A^T + \vec{u}[0]^T B^T + \vec{e}[0]^T$$
$$\vec{x}[2]^T = \vec{x}[1]^T A^T + \vec{u}[1]^T B^T + \vec{e}[1]^T$$
$$\vdots$$
$$\vec{x}[l]^T = \vec{x}[l-1]^T A^T + \vec{u}[l-1]^T B^T + \vec{e}[l-1]^T$$

$$\underbrace{\begin{bmatrix} \vec{x}[1]^T \\ \vdots \\ \vec{x}[l]^T \end{bmatrix}}_{S_{l\times n}} = \underbrace{\begin{bmatrix} \vec{x}[0]^T & \vec{u}[0]^T \\ \vdots & \\ \vec{x}[l-1]^T & \vec{u}[l-1]^T \end{bmatrix}}_{D_{l\times(n+m)}} \underbrace{\begin{bmatrix} A^T \\ B^T \end{bmatrix}}_{P_{(n+m)\times n}} + \underbrace{\begin{bmatrix} \vec{e}[0]^T \\ \vdots \\ \vec{e}[l-1]^T \end{bmatrix}}_{E}$$

System Identification

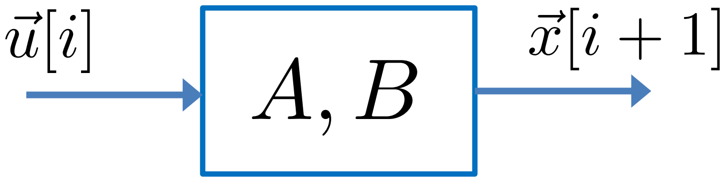
Problem: consider the discrete linear time invariant system:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$$

Given: observed inputs and outputs:

$$\vec{u}[0], \vec{u}[1], \ldots, \vec{u}[l], \ldots$$
$$\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[l], \ldots$$

Objective: learn the system parameters:



Vector Case: $\vec{x} \in \mathbb{R}^n, \vec{e} \in \mathbb{R}^n, \vec{u} \in \mathbb{R}^m,$
$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}.$

$$S = \underbrace{DP} + E$$

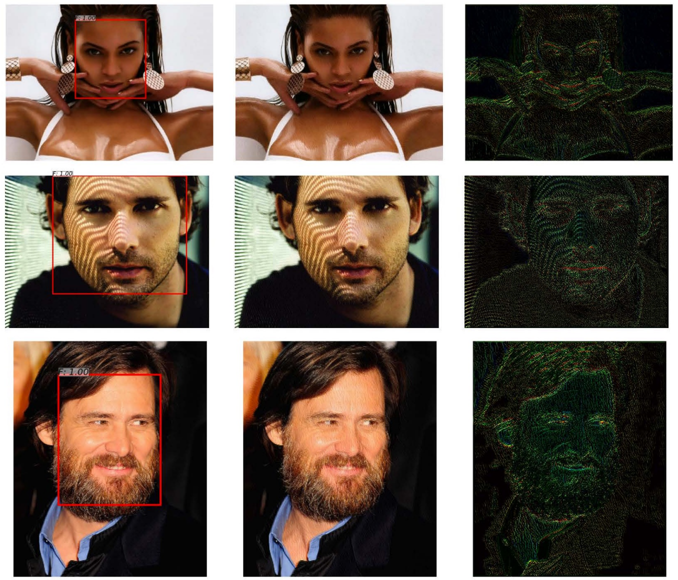
$$P_* = (D^T D)^{-1} D^T S.$$

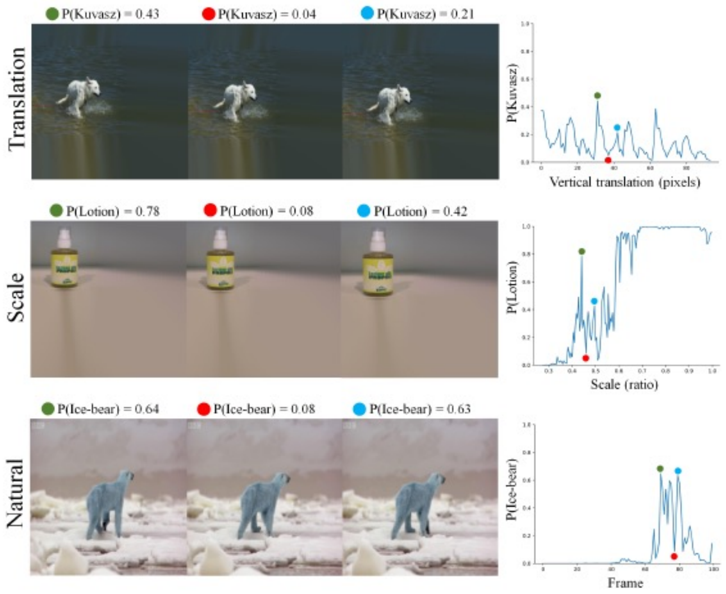
$$E_* = S - DP_*$$
$$= S - D(D^T D)^{-1} D^T S$$
$$= (I - D(D^T D)^{-1} D^T) S$$

System Stability (and Robustness)

$$\vec{u}[i] \longrightarrow \boxed{A, B} \longrightarrow \vec{x}[i+1]$$

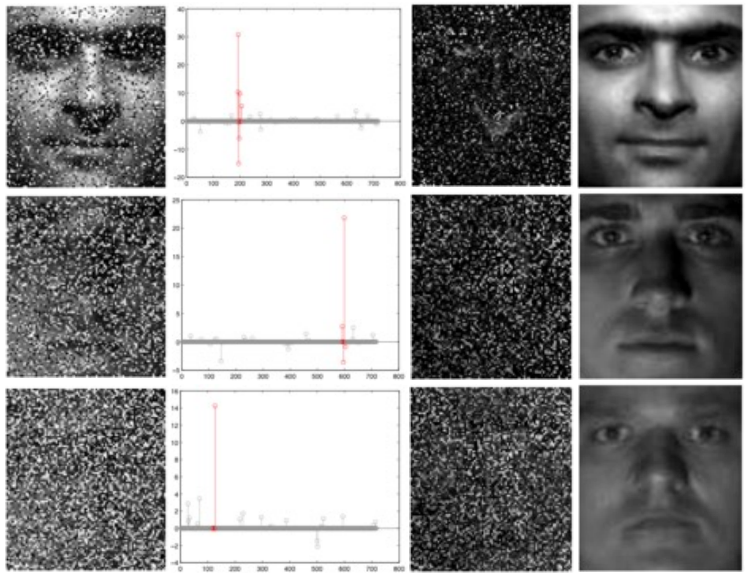
Stability (or lack thereof)?





Modern deep neural networks
for face detection or object recognition

Robustness to corruption & attack



Robust face recognition
Wright & Ma 2008

System Stability

Scalar Case: $x[i+1] = \lambda x[i] + u[i] + e[i]$ (with $u[i] = 0$)

$x[i+1] = \lambda x[i]$ (with $\lambda > 1$)

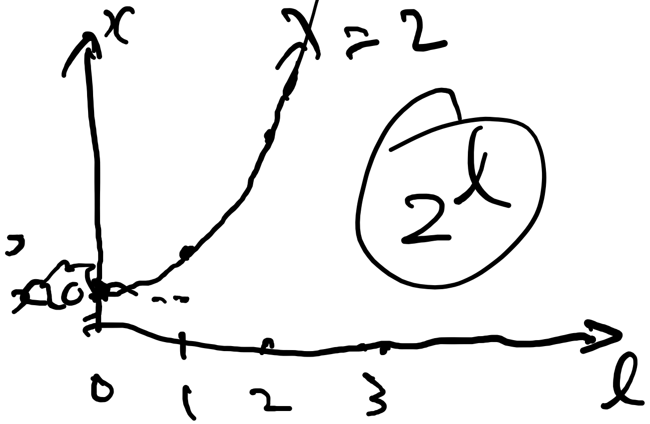
$x[1] = \lambda x[0]$

$x[2] = \lambda x[1] = \lambda^2 x[0]$

$\vdots$

$x[\ell] = \lambda x[\ell-1] = \lambda^\ell x[0]$

$\lambda = 2$



2^ℓ

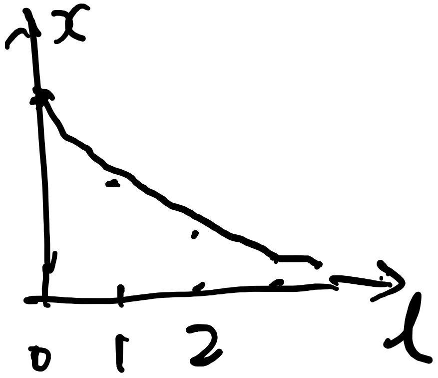
$\lambda > 1$

$(1+\varepsilon)^\ell$

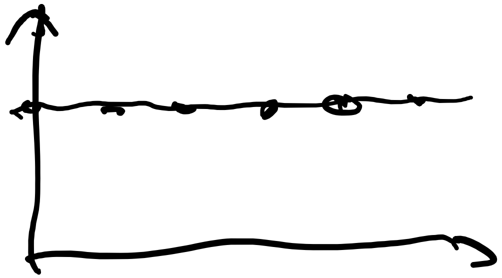
$x[i+1] = \lambda x[i]$ (with $\lambda \leq 1$)

$\lambda = 1/2$

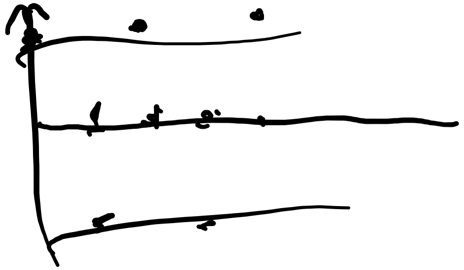
$x[\ell] = \left(\frac{1}{2}\right)^\ell x[0]$



$\lambda = 1$

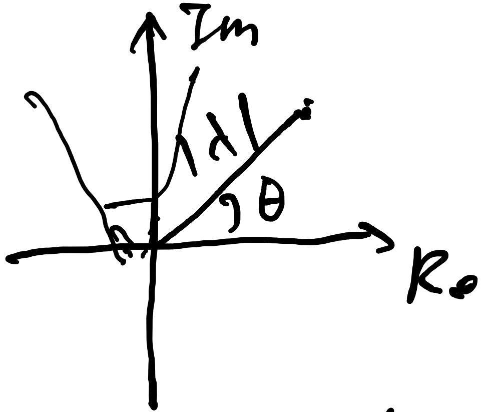


$\lambda = -1$



System Stability

Complex λ: $x[i+1] = \lambda x[i] \quad (\text{with } \lambda = |\lambda| e^{j\theta})$



$$x[1] = \lambda x[0]$$

$$\vdots$$

$$x[l] = \lambda^{l} x[0]$$

$$= (|\lambda| e^{j\theta})^{l} x[0]$$

$$= |\lambda|^{l} e^{jl\theta} x[0]$$

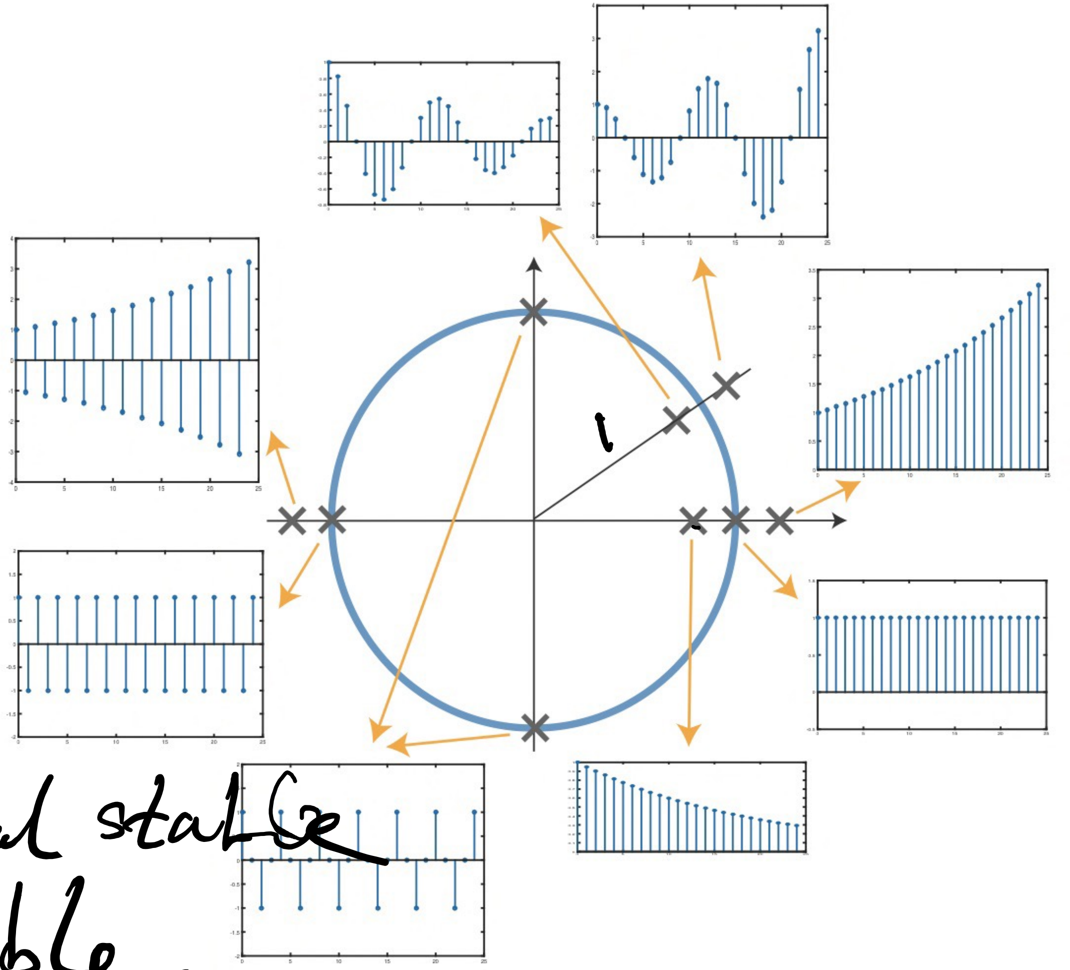
$$|x[l]| = |\lambda|^{l} \underbrace{|e^{jl\theta}|}_{1} |x[0]|$$

$$= |\lambda|^{l} |x[0]|$$

$|\lambda| < 1$ stable

$|\lambda| \leq 1$ critical stable

$|\lambda| > 1$ unstable.



System Stability (with Input)

$$x[i+1] = \lambda x[i] + e[i]$$

Critical Case $|\lambda| = 1$**:**

Bounded Input Bounded State Stability

Definition: We say a system is *bounded input bounded state (BIBS) stable* if its state stays bounded, $\forall i \; \|\vec{x}[i]\| \leq C$, for any initial condition, any bounded input, and bounded disturbance.

$$x[i+1] = \lambda x[i] + u[i] + e[i] \;\in \mathbb{R} \qquad\qquad \vec{x}[i+1] = A\vec{x}[i] + \vec{u}[i] + \vec{e}[i] \;\in \mathbb{R}^n$$

When is the above scalar system stable by this definition?

What about the vector case?