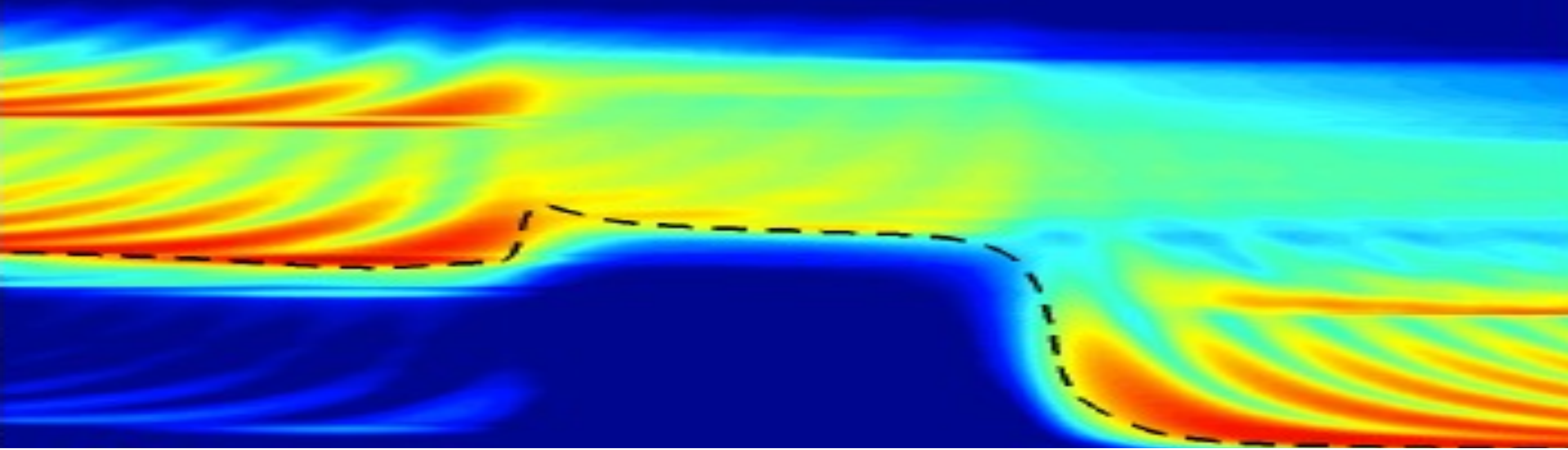

EECS 16B
Designing Information Devices and Systems II
Lecture 14

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Outline

- System BIBS Stability (scalar case)
- System BIBS Stability (vector case)
- Stability of Continuous Time Systems

System Stability (Recap)

Definition: We say a system is *bounded input bounded state (BIBS) stable* if its state stays bounded, $\forall i \; \|\vec{x}[i]\| \leq C$, for any initial condition, any bounded input, and bounded disturbance.

Scalar Case : $x[i+1] = \lambda x[i] + e[i], \quad |\lambda| = 1$

$\lambda > 1 \qquad x[i] \uparrow$ as $i \to \infty$

λ^{l}.

$\rightarrow$ bounded

$x[1] = 1 x[0] + e[0]$

$x[2] = x[0] + e[1] + e[0] \qquad e[i] = M$

$\vdots$

$x[l] = x[0] + \underbrace{e[l-1] + \cdots + e[0]}_{l \cdot M \uparrow \infty}$

$|\lambda| < 1$?

System Stability

Stability for the Scalar Case: $x[i+1] = \lambda x[i] + e[i] \quad (|\lambda| < 1)$

Claim: If $|\lambda| < 1$, then for any $x[0]$ and bounded $|e[i]| < M$ the solution to the system $x[i+1] = \lambda x[i] + e[i]$ remains bounded.

Proof:

$x[1] = \lambda x[0] + e[0]$

$x[2] = \lambda x[1] + e[1] = \lambda^2 x[0] + e[1] + \lambda e[0]$

$\vdots$

$x[\ell] = \lambda^\ell x[0] + \sum_{k=0}^{\ell-1} \lambda^k e[\ell-1-k]$

$\lambda^\ell x[0] \searrow 0$ as $\ell \to \infty$

$(1 + t + t^2 + t^3 + \cdots) = \frac{1}{1-t}$, $\quad t < 1$

$t = |\lambda|$

$\left| \sum_{k=0}^{\ell-1} \lambda^k e[\ell-1-k] \right| \leq \tilde{M}$

$\leq \sum_{k=0}^{\ell-1} |\lambda|^k \, |e[\ell-1-k]|$

$\leq M \sum_{k=0}^{\ell-1} |\lambda|^k \quad \leq \ell \cdot M$ as $\ell \nearrow$

$\leq M \cdot \frac{1}{1-|\lambda|}$

$\leq \tilde{M}$

$\blacksquare$

System Stability

Stability for the Scalar Case: $x[i+1] = \lambda x[i] + e[i] \quad (|\lambda| < 1)$

Claim: If $|\lambda| < 1$, then for any $x[0]$ and bounded $|e[i]| < M$ the solution to the system $x[i+1] = \lambda x[i] + e[i]$ remains bounded.

Proof:

System Stability

$A_{n\times n}$

Stability for the Vector Case: $\vec{x}[i+1] = A\vec{x}[i] + \vec{e}[i] \in \mathbb{R}^n$

$\vec{x}[1] = A\vec{x}[0] + \vec{e}[0]$

$\vec{x}[2] = A\vec{x}[1] + \vec{e}[1] = A^2\vec{x}[0] + \vec{e}[1] + A\vec{e}[0]$

$\vdots$

$\vec{x}[\ell] = A^\ell \vec{x}[0] + \sum_{k=0}^{\ell-1} A^k \vec{e}[\ell-1-k]$

$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

$A^\ell = \underbrace{A \cdot A \cdots A}_{\ell}.$

$|\vec{e}_k[i]| \le M$

$|\vec{x}[i]|$? bounded. when?

A ?

System Stability

Stability for the Vector Case: $\vec{x}[i+1] = A\vec{x}[i] + \vec{e}[i] \in \mathbb{R}^n$

Diagonalizable: $\Lambda = V^{-1}AV$ $\quad \vec{z} = V^{-1}\vec{x}$ $\quad \vec{x} = V\vec{z}$

$$\begin{bmatrix} 2 \\ 3 & 4 \end{bmatrix}$$

eigenvalues

$$V\Lambda = AV$$

$$\underbrace{A[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]}_{V} = \underbrace{[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]}_{V}\underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}_{\Lambda}$$

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$\vec{z}[i+1] = V^{-1}\vec{x}[i+1] = V^{-1}(A\underbrace{V\vec{z}[i]}_{\vec{x}[i]} + \cancel{V}\vec{e}[i])$$

$$\vec{z}[i+1] = \Lambda\vec{z}[i] + V^{-1}\vec{e}[i]$$

$$\lambda_1 = 2, \; z_1 \uparrow$$

$$\vec{x}[i] = V\vec{z}[i]$$

$$\begin{bmatrix} z_1[i+1] \\ \vdots \\ z_n[i+1] \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}\begin{bmatrix} z_1[i] \\ \vdots \\ z_n[i] \end{bmatrix} + V^{-1}\vec{e}[i]$$

$$z_k[i+1] = \lambda_k z_k[i] + (V^{-1}\vec{e}[i])_k$$

BIBS stable iff

$|\lambda_k| < 1$ for all

$i = 1, 2, \ldots, n$

System Stability

Stability for the Vector Case: $\vec{x}[i+1] = A\vec{x}[i] + \vec{e}[i] \in \mathbb{R}^n$

Triangularizable: $T = V^{-1}AV \quad \vec{z} = V^{-1}\vec{x}$

A cannot be diagonalized?

$$T = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ & \lambda_2 & & \vdots \\ & & \ddots & * \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\vec{z}[i+1] = V^{-1}\vec{x}[i+1] = V^{-1}AV\vec{z}[i] + V^{-1}\vec{e}[i]$$

$$\begin{bmatrix} z_1[i+1] \\ z_2[i+1] \\ \vdots \\ z_n[i+1] \end{bmatrix} = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ & \lambda_2 & & \vdots \\ 0 & & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} z_1[i] \\ z_2[i] \\ \vdots \\ z_n[i] \end{bmatrix} + V^{-1}\vec{e}[i]$$

$$\det(\lambda I - T)$$

$$\begin{bmatrix} \lambda-\lambda_1 & * & & * \\ & \lambda-\lambda_2 & & \\ 0 & & \ddots & * \\ & & & \lambda-\lambda_n \end{bmatrix}$$

$$\det(\lambda I - T) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

$$z_n[i+1] = \lambda_n z_n[i] + (V^{-1}\vec{e}[i])_n$$

$|\lambda_n| < 1 \quad z_n[i]$ bounded.

$$z_{n-1}[i+1] = \lambda_{n-1} z_{n-1}[i] + \boxed{* \, z_n[i] + \left(V^{-1} \vec{e}[i]\right)_{n-1}} \quad \text{bounded}$$

$z_{n-1}[i]$ bounded if $|\lambda_{n-1}| < 1$

by induction system is BIBS stable iff

all $|\lambda_k| < 1$ $\quad k = 1, 2, \ldots, n.$ $\square$

System Stability (Discrete Time)

$|\lambda_k|^{\ell}$

Stability for the Vector Case: $\vec{x}[i+1] = A\vec{x}[i] + \vec{e}[i] \in \mathbb{R}^n$

Summary: The above discrete time system is BIBS stable if all eigenvalues λ_k of A satisfy:

$$|\lambda_k| < 1, \quad k = 1, \ldots, n.$$

That is, all eigenvalues are strictly inside the unit circle in the complex plane.

$\lambda_1 = 0$

$A = \begin{bmatrix} 0.5 & 3 \\ 0 & -0.5 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

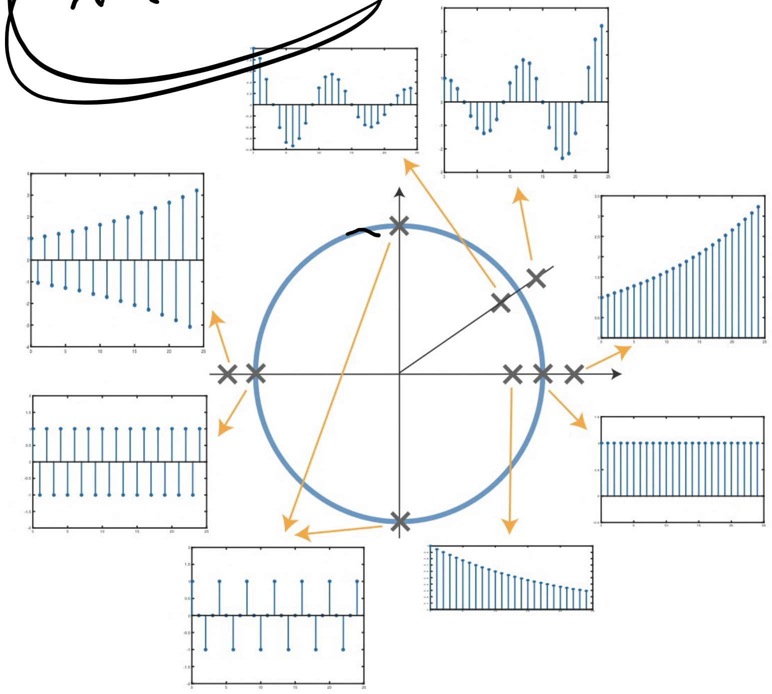
$\lambda_2 = 2$

$A \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$\lambda_1 = 0.5$

$\lambda_2 = 0$

$\lambda_3 = 2$



System Stability (Continuous Time)

Stability for the Scalar Case: $$\frac{d}{dt}x(t) = \lambda x(t) + w(t)$$

$$\frac{x[i+1] - x[i]}{\Delta} = \lambda x[i] + w[i]$$

$$x(t) = e^{\lambda t}x(0) + \int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau$$

$$x[i+1] = (1+\lambda\Delta)x[i] + \Delta w[i]$$

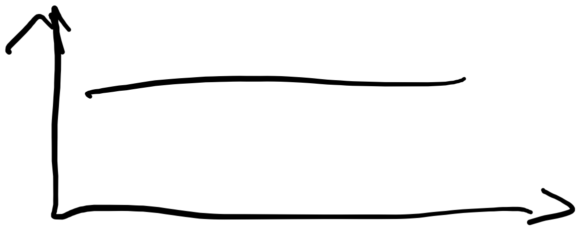
$$\left|\int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau\right| \leq \int_0^t e^{\lambda(t-\tau)}d\tau M = \frac{e^{\lambda t}-1}{\lambda}M$$

$$|(1+\lambda\Delta)| < 1$$

$e^{\lambda t}$ bounded?

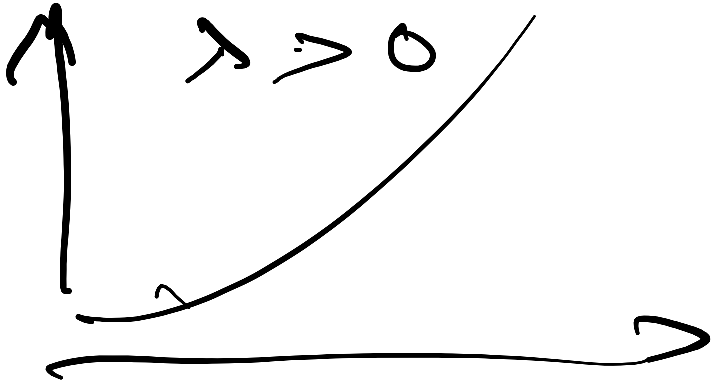
$\lambda < 0$

$\lambda = 0$



$\lambda < 0$

$\lambda > 0$



System Stability (Continuous Time)

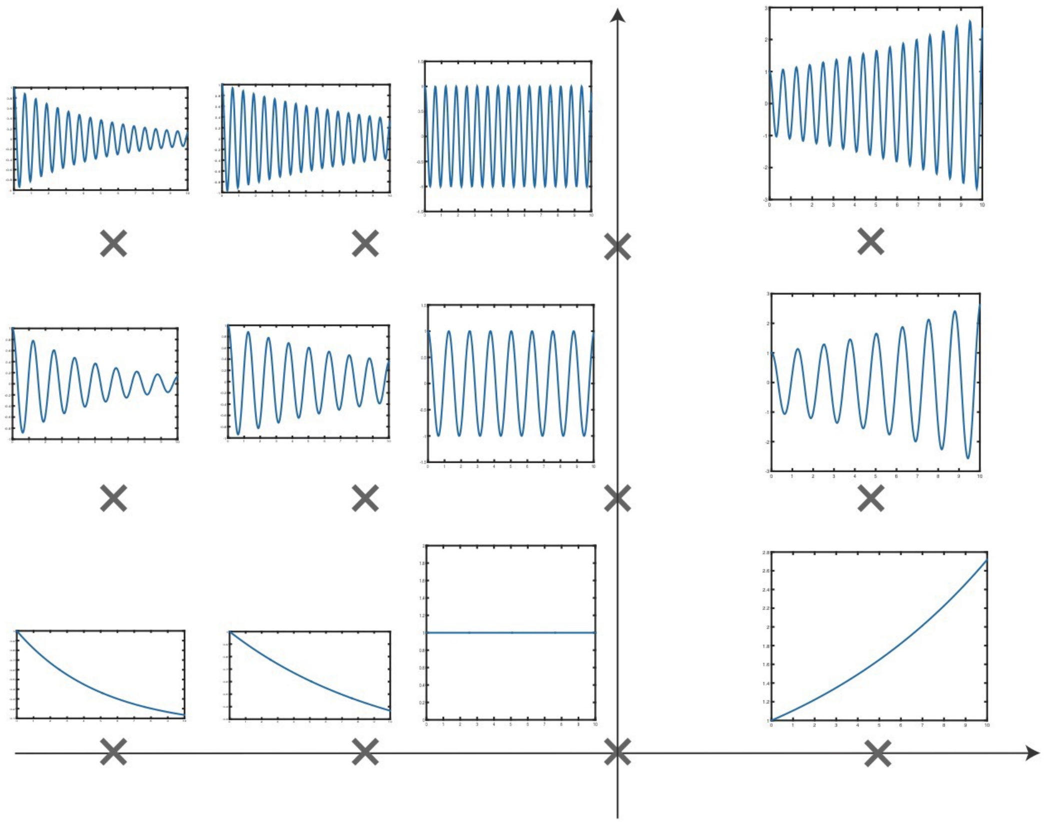
Stability for the Scalar Case: $\frac{d}{dt}x(t) = \lambda x(t) + w(t)$

$\lambda = \alpha + j\theta$

$e^{\lambda t} = e^{(\alpha + j\theta)t}$

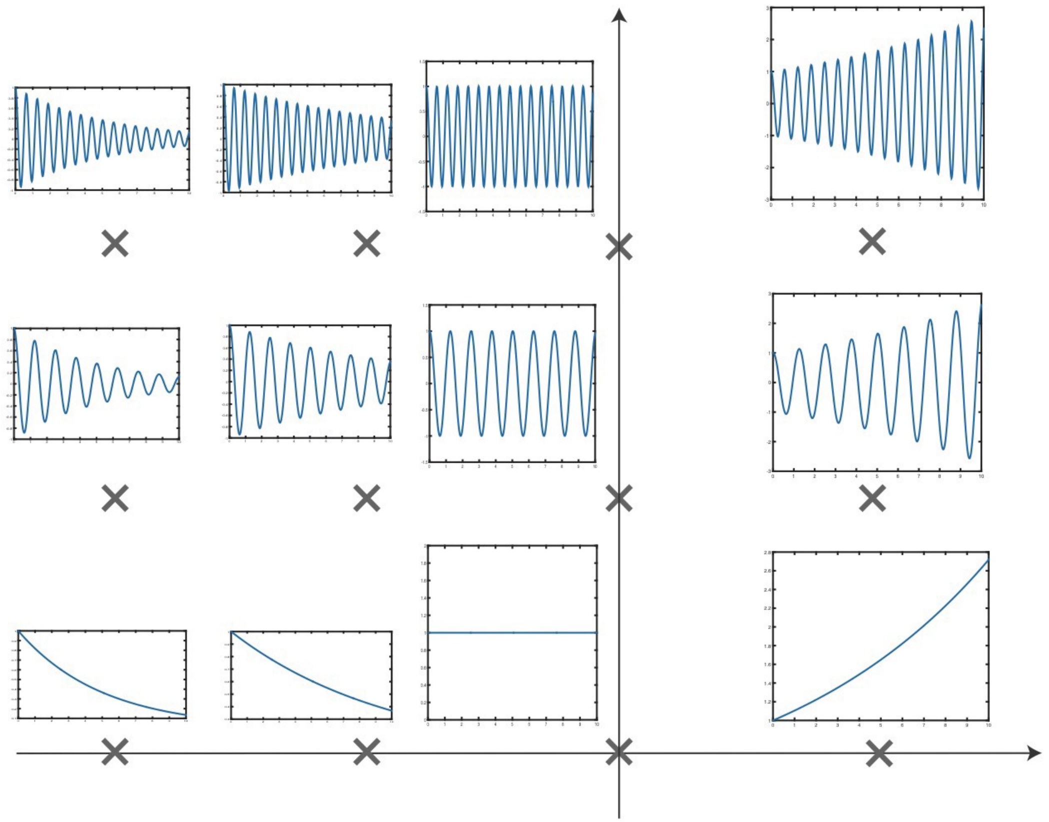
$= \left| e^{\alpha t} \cdot \underline{e^{j\theta t}} \right|$

$= e^{\alpha t} \qquad \alpha < 0$



System Stability (Continuous Time)

Stability for the Scalar Case: $$\frac{d}{dt}x(t) = \lambda x(t) + w(t)$$



System Stability (Continuous Time)

Stability for the Vector Case: $\dot{\vec{x}}(t) = A\vec{x}(t) + \vec{w}(t) \;\; \in \mathbb{R}^n$

Diagonalize or triangularize: $T = V^{-1}AV \quad \vec{z} = V^{-1}\vec{x}$

System Stability (Summary)

Continuous time: $\dot{\vec{x}}(t) = A\vec{x}(t) + \vec{w}(t) \ \in \mathbb{R}^n$

Discrete time: $\vec{x}[i+1] = A\vec{x}[i] + \vec{e}[i] \ \in \mathbb{R}^n$