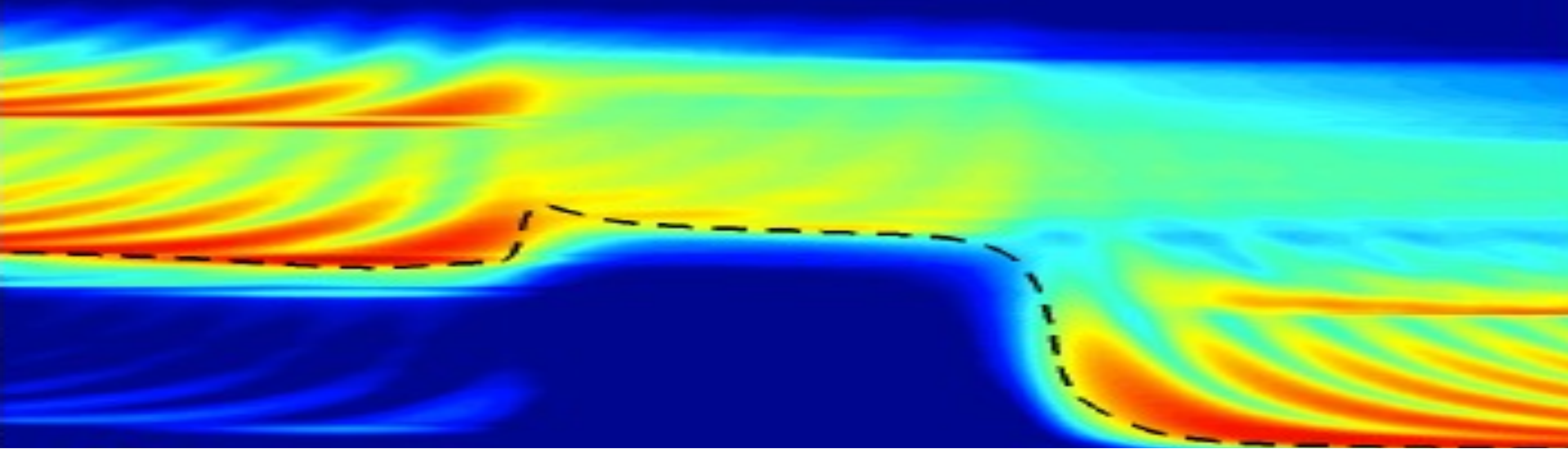

EECS 16B
Designing Information Devices and Systems II
Lecture 15

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Outline

- System Stability (Recap)
- Stabilization by Feedback
- Control Canonical Form

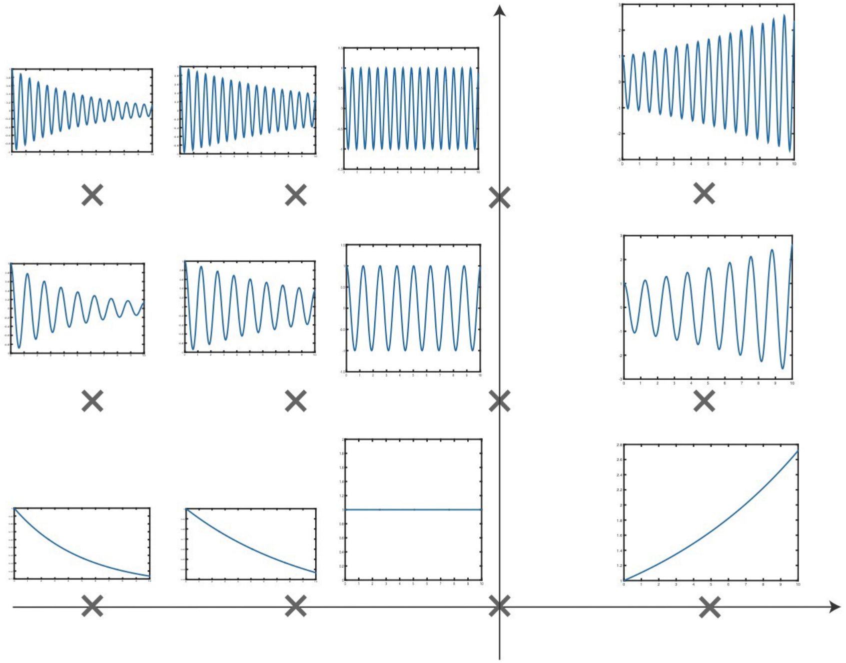
System Stability (Continuous Time)

Stability for the Scalar Case: $\frac{d}{dt}x(t) = \lambda x(t) + w(t)$

$$\frac{x[i+1] - x[i]}{\Delta} = \lambda x[i] + w[i]$$

$$x(t) = e^{\lambda t}x(0) + \int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau$$

$$|\int_0^t e^{\lambda(t-\tau)}w(\tau)d\tau| \leq \int_0^t e^{\lambda(t-\tau)}d\tau M = \frac{e^{\lambda t} - 1}{\lambda}M$$



System Stability (Continuous Time)

Stability for the Vector Case: $\dot{\vec{x}}(t) = A\vec{x}(t) + \vec{w}(t) \;\; \in \mathbb{R}^n$

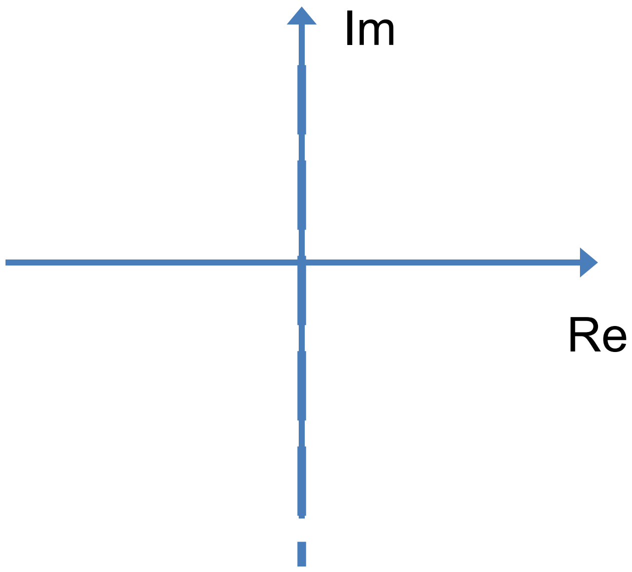
Diagonalize or triangularize: $T = V^{-1}AV \quad \vec{z} = V^{-1}\vec{x}$

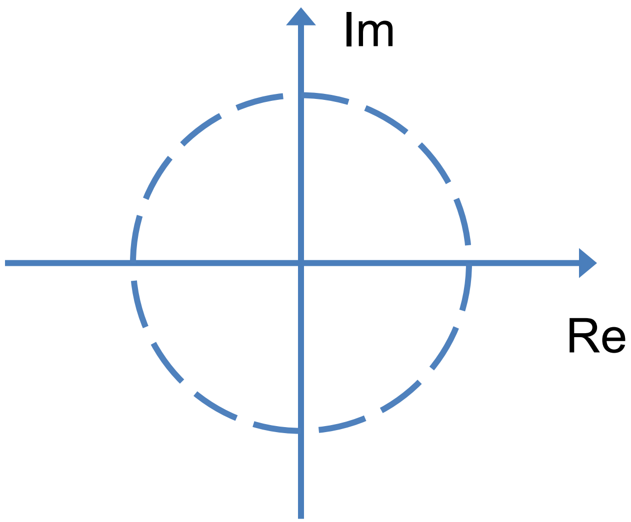
System Stability (Recap)

Definition: We say a system is *bounded input bounded state (BIBS) stable* if its state stays bounded, $\forall i \ \|\vec{x}[i]\| \leq C$, for any initial condition, any bounded input, and bounded disturbance.

Continuous time: $\dot{\vec{x}}(t) = A\vec{x}(t) + \vec{w}(t) \ \in \mathbb{R}^n$

Discrete time: $\vec{x}[i+1] = A\vec{x}[i] + \vec{e}[i] \ \in \mathbb{R}^n$

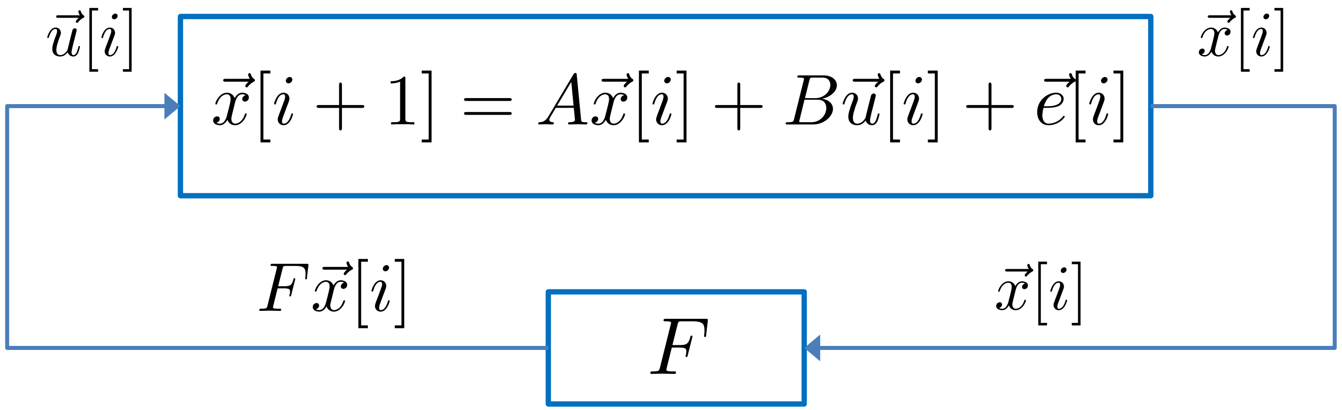




System Stabilization

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i] \;\in \mathbb{R}^n$$

What if some or all eigenvalues of A are outside of the unit circle? Consider the feedback: $\vec{u}[i] = F\vec{x}[i]$



$$A_{cl} = A + BF$$

System Stabilization (Example 1)

Scalar case: $x[i+1] = 3x[i] + u[i] + e[i]$

System Stabilization (Example 2)

Vector case: $\vec{x}[i+1] = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{e}[i]$

System Stabilization (Example 3)

Vector case: $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] + \vec{e}[i]$

System Stabilization (Example 3)

Vector case: $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] + \vec{e}[i]$

Controllable Canonical Form

Single input case: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$ $\qquad A_{cl} = A + BF$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$F = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \end{bmatrix}$$

Controllable Canonical Form

Characteristic Polynomial of A is simple:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & -1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^n - a_n \lambda^{n-1} - a_{n-1} \lambda a^{n-2} - \cdots - a_2 \lambda - a_1$$

Controllable Canonical Form

$$\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n \qquad A_{cl} = A + BF$$

$$A_{cl} = A + BF = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \end{bmatrix}$$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

Can we bring the system to the canonical form via a similarity transform: $\vec{z} = T\vec{x}$?

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

Claim: we can convert the above system to the canonical form if the following **controllability matrix:**

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots | AB \mid B] \ \in \mathbb{R}^{n\times n} \text{ is invertible.}$$

$$\vec{z}[i+1] = TAT^{-1}\vec{z}[i] + TBu[i] + T\vec{e}[i] \ \in \mathbb{R}^n \qquad \vec{z} = T\vec{x}$$

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a'_1 & a'_2 & \cdots & a'_{n-1} & a'_n \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \ \in \mathbb{R}^n$

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$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a'_1 & a'_2 & \cdots & a'_{n-1} & a'_n \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Controllable Canonical Form

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \;\; \in \mathbb{R}^n$

$\vec{z}[i+1] = TAT^{-1}\vec{z}[i] + TBu[i] + T\vec{e}[i] \;\; \in \mathbb{R}^n \qquad \vec{z} = T\vec{x}$

$\vec{z}[i+1] = A_z\vec{x}[i] + B_z u[i] + \vec{e}'[i]$

$u[i] = F_z\vec{z}[i] = F_z T\vec{x}[i]$

Claim: the closed loop system $A + BF = A + BF_zT$ has the same eigenvalues as $A_z + B_zF_z$

Feedback Control (Summary)

For a general system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i] \;\in \mathbb{R}^n$

- It is possible to stabilize the system via state **feedback control**:

$$\vec{u}[i] = F\vec{x}[i]$$

- When is this possible? The system is **controllable**:

$$\mathcal{C} \doteq [A^{n-1}B \mid \cdots \mid AB \mid B] \;\in \mathbb{R}^{n\times n} \text{ is invertible.}$$

- How to design eigenvalues of closed-loop system (to stabilize)? Controllable **canonical form**:

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ a'_1 & a'_2 & \cdots & a'_{n-1} & a'_n \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$